Chapter 3 - Projectiles

Section 3A: Vectors

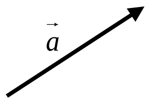
A **vector** is a quantity which has both a magnitude and a direction, e.g. the velocity of the wind was 15 m s^{-1} towards the north. Velocity is an example of a vector quantity.

A **scalar** is a quantity which has magnitude only, e.g. the car travelled along the road with a speed of 20 m s^{-1}. Speed is an example of a scalar quantity.

Examples of vector and scalar quantities

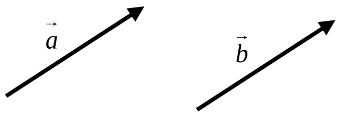
Vector Quantities	Scalar Quantities
Displacement	Length
Velocity	Area
Acceleration	Volume
Momentum	Speed
Force	Time
	Density
	Mass
	Energy

Vectors are usually represented in diagrams by **drawing an arrow**, where the length of the arrow represents the magnitude (size) of the vector, and the direction in which it is drawn represents the direction in which the vector acts.



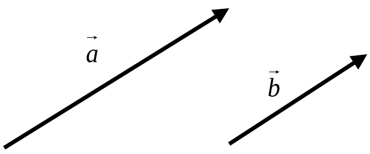
Note: A vector is marked as a vector quantity by putting an arrow over it, like the $\vec{a}$ above. This is to distinguish it from a scalar quantity. In some cases instead of putting the arrow over a vector quantity, a squiggly line is put underneath or they are printed in bold type. It is important that vectors are marked in some way. A vector that goes from point A to point B could be shown as $\overrightarrow{AB}$.

Example 1) For two vectors to be **equal**, they must have the same magnitude and direction, not just the same magnitude.



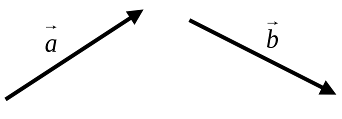
Vectors $\vec{a}$ and $\vec{b}$ are exactly **equal**, as both their magnitudes and directions are the same.
$\therefore \vec{a} = \vec{b}$

Example 2) If two vectors are **parallel**, they have the same direction, but different magnitudes, and one vector is a multiple of the other.



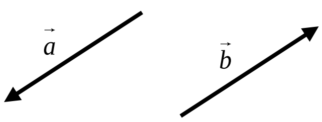
Vectors $\vec{a}$ and $\vec{b}$ are **parallel**, as both their directions are the same, but their magnitudes are different.
$\therefore \vec{a} = k\vec{b}$, where k is a scalar quantity (number).

Example 3) If two vectors have **the same magnitude**, but their directions are different, they are not equal.



Vectors $\vec{a}$ and $\vec{b}$ have the **same magnitude**, but their directions are different.
$\therefore \vec{a} \neq \vec{b}$, but $|\vec{a}| = |\vec{b}|$,
where $|\vec{a}|$ means the magnitude (length) of $\vec{a}$.

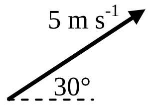
Example 4) If two vectors have the same magnitude and their directions are exactly opposite to each other, the two vectors are **equal and opposite.**

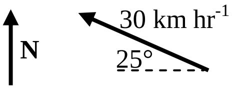


Vectors $\vec{a}$ and $\vec{b}$ are **equal and opposite**, as both their magnitudes are the same, and their directions are exactly opposite each other.
$\therefore \vec{a} = -\vec{b}$.

Vectors are generally expressed in two different common forms: **magnitude and direction form**, where the size and direction of the vector are given in some way directly, or in **Cartesian form** ($\vec{i}-\vec{j}$ form), where the vector is described in terms of two orthogonal (perpendicular) unit vectors, as described below.

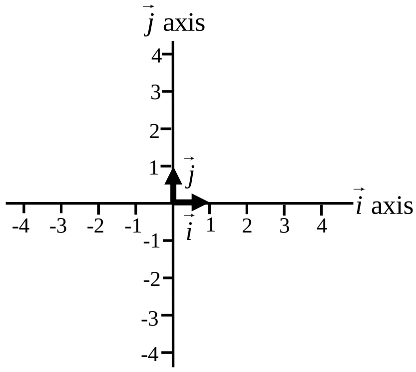
Magnitude and Direction form: an example of a vector in magnitude and direction form is: a velocity of 5 m s^{-1} at an angle 30° above the horizontal. In this case the magnitude is 5 m s^{-1}, and the direction is 30° above the horizontal.

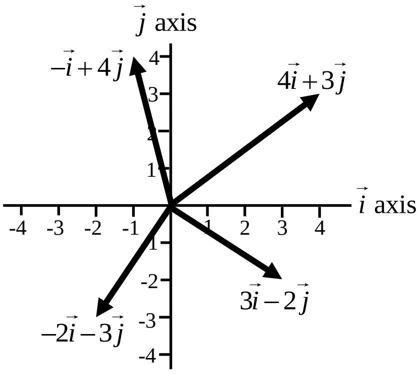




An alternative way in which a vector might be expressed in magnitude and velocity form is: a velocity of 30 km hr^{-1} in a direction 25° North of West. In this case the direction is given in relation to the cardinal points.

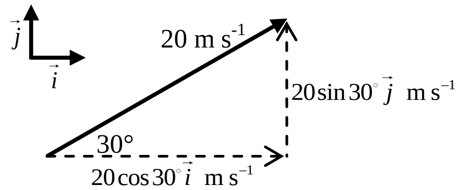
Cartesian form ($\vec{i}-\vec{j}$ form): A convenient way for doing calculations involving vectors is to convert them to Cartesian form, instead of Magnitude and Direction form. In Cartesian form, any vector can be expressed in terms of $\vec{i}$ and $\vec{j}$, which are two orthogonal (perpendicular) unit vectors. Usually the $\vec{i}$ direction is taken along the *x*-axis and the $\vec{j}$ direction is taken along the *y*-axis, as shown in the diagram. Any vector can then be written as a combination of these.





The diagram to the left shows four different vectors expressed in $\vec{i}$ - $\vec{j}$ form. Any vector can be expressed in this way.

To convert a vector in Magnitude and Direction form to Cartesian form: If we have a vector, such as a velocity, given in magnitude and direction form as 20 m s^{-1} in the direction 30° North of East, and we wish to convert it to Cartesian form. We will assume that the $\vec{i}$ direction is East and the $\vec{j}$ direction is North.

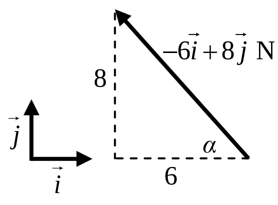


The vector must be broken down into two perpendicular components, one horizontal and the other vertical, as shown in the diagram. **This is called resolving a vector into its components**.

The sine and cosine of the angle are used to find these components.
The magnitude of the vector (20 m s^{-1}) is the hypotenuse of the triangle, the $\vec{i}$ component is the adjacent side and the $\vec{j}$ component is the side opposite the angle. Since the cosine of the angle relates the adjacent side to the hypotenuse, then the $\vec{i}$ component is $20\cos 30^\circ \vec{i}$ m s^{-1}. Similarly since the sine of the angle relates the opposite side to the hypotenuse, then the $\vec{j}$ component is $20\sin 30^\circ \vec{j}$ m s^{-1}. Overall then the velocity vector is written in Cartesian form as: $20\cos 30^\circ \vec{i}+20\sin 30^\circ \vec{j}$ m s^{-1}. Using the values for the sine and cosine of 30° from section 0A, it is written: $20\left(\frac{\sqrt{3}}{2}\right)\vec{i}+20\left(\frac{1}{2}\right)\vec{j}=10\sqrt{3}\vec{i}+10\vec{j}$ m s^{-1}.

Therefore a velocity of 20 m s^{-1} in a direction 30° North of East is $10\sqrt{3}\vec{i}+10\vec{j}$ m s^{-1} in Cartesian form.

To convert a vector in Cartesian form to Magnitude and Direction form
If we have a vector, such as a force, given in Cartesian form as $-6\vec{i}+8\vec{j}$ N, and we wish to convert it to magnitude and direction form. To find the magnitude of the vector, this is called the **modulus** of the vector, and is found using Pythagoras' Theorem. If we consider the $\vec{i}$ and $\vec{j}$ components as two sides of a right-angled



triangle, then the hypotenuse is the magnitude of the vector. So in this example, the magnitude of the force vector is $\sqrt{(-6)^2+8^2}=10$ N. To find the direction of the force vector, we must find the angle (α) it makes with the horizontal direction. From the right-angled triangle shown in the diagram it can be seen that $\tan\alpha=\frac{8}{6}=\frac{4}{3}$. Therefore the angle α is given by $\tan^{-1}\frac{4}{3}=53.1^\circ$. Therefore a force of $-6\vec{i}+8\vec{j}$ N is the same as a force of 10 N in the direction 53.1° North of West.

Finding the resultant of two or more vectors in Cartesian form

The **resultant** of two or more vectors is found by adding the vectors together. For example the resultant of a set of different forces acting on a body is found by adding all the forces together. The resultant is not simply the sum of all the magnitudes of the vectors, as this does not take their directions into account. When vectors are in Cartesian form, the resultant is found by simply adding all the $\vec{i}$ components together and adding all the $\vec{j}$ components together.

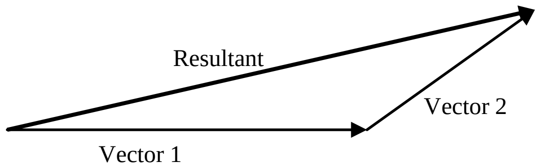
E.g. $\vec{a}=4\vec{i}-3\vec{j},\qquad \vec{b}=-\vec{i}+5\vec{j}$

$$\vec{a}+\vec{b}=4\vec{i}-3\vec{j}-\vec{i}+5\vec{j}=(4-1)\vec{i}+(-3+5)\vec{j}=3\vec{i}+2\vec{j}$$

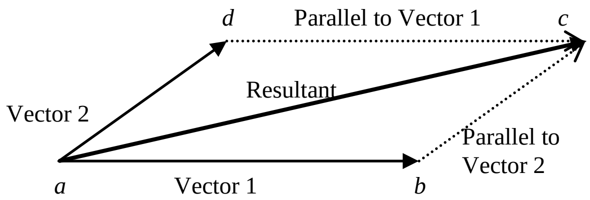
Finding the resultant of two or more vectors in magnitude and direction form

When two vectors are in magnitude and resultant form and their resultant is required, one method is to convert both vectors to Cartesian form, and find the resultant as shown above. Otherwise the resultant can be found by drawing the triangle of vectors, or the parallelogram of vectors.

Triangle Law for finding the resultant of two vectors: If 2 vectors are drawn head to tail, the vector from the tail of the first to the head of the second is the resultant.



Parallelogram Law for finding the resultant of two vectors: If 2 vectors, drawn tail to tail, are the adjacent sides *ab* and *ad* of a parallelogram *abcd*, the diagonal *ac* is the resultant.

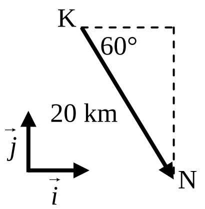


In reality in Applied Mathematics we usually do not need to draw an accurate version of either the triangle of vectors or of the parallelogram of vectors. Usually either they are converted to Cartesian form to find the resultant, or an approximate diagram of the triangle of vectors is drawn and from this the unknown sides of the triangle and unknown angles can be calculated.

Example 3A1

The displacement of the town of Newtown from the town of Kilderry is 20 km in the direction 60° South of East. Express this displacement as a vector in Cartesian form, where the $\vec{i}$ direction is East and the $\vec{j}$ direction is North.

Solution:



$$\overrightarrow{KN}=20\cos60^\circ\vec{i}-20\sin60^\circ\vec{j}$$

$$\Rightarrow \overrightarrow{KN}=20\left(\frac{1}{2}\right)\vec{i}-20\left(\frac{\sqrt{3}}{2}\right)\vec{j}$$

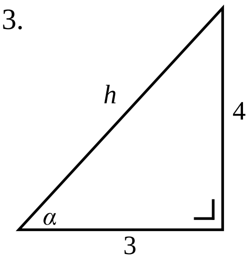
$$\Rightarrow \overrightarrow{KN}=10\vec{i}-10\sqrt{3}\vec{j}\text{ km}$$

Example 3A2

If $\tan\alpha = \dfrac{4}{3}$, write $\sin\alpha$ and $\cos\alpha$ as fractions, using surds where necessary.

Solution:

The first step is to draw a right-angled triangle, putting in the known sides, and from these calculate the third side using Pythagoras' Theorem. In this case the tan of the angle is $\frac{4}{3}$, so the opposite is 4 and the adjacent is 3.



$\tan\alpha = \dfrac{4}{3}$. Therefore the opposite is 4 and the adjacent is 3.

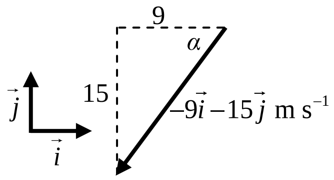
By Pythagoras' Theorem: $h^2 = 4^2 + 3^2 = 16 + 9 = 25, \quad \Rightarrow h = \sqrt{25} = 5$

$\sin\alpha = \dfrac{o}{h} = \dfrac{4}{5}, \qquad \cos\alpha = \dfrac{a}{h} = \dfrac{3}{5}.$

Example 3A3

The velocity of a car is expressed as $-9\vec{i} - 15\vec{j}$ m s^{-1}. Find the actual speed of the car and the direction in which it is travelling, if the $\vec{i}$ direction is East and the $\vec{j}$ direction is North.

Solution:



$\vec{v} = -9\vec{i} - 15\vec{j}$ m s^{-1}

By Pythagoras' Theorem: $\left|\vec{v}\right|^2 = (-9)^2 + (-15)^2 = 81 + 225 = 306$

$\Rightarrow \left|\vec{v}\right| = \sqrt{306} = 17.5$ m s^{-1}

$\tan\alpha = \dfrac{15}{9} = \dfrac{5}{3}, \qquad \Rightarrow \alpha = \tan^{-1}\dfrac{5}{3} = 59.0^\circ$

Therefore the car travels at 17.5 m s^{-1} in the direction 59.0° South of West.

Example 3A4

$\vec{a} = 4\vec{i} + 3\vec{j} \qquad \vec{b} = -2\vec{i} + \vec{j} \qquad \vec{c} = \vec{i} - 5\vec{j}$

Find the following resultants in $\vec{i} - \vec{j}$ form:

(i) $\vec{a} + \vec{b} + \vec{c}$, (ii) $2\vec{a} - 5\vec{b}$, (iii) $\vec{a} + 2\vec{b} - 3\vec{c}$.

Solution:

(i) $\vec{a} + \vec{b} + \vec{c} = 4\vec{i} + 3\vec{j} - 2\vec{i} + \vec{j} + \vec{i} - 5\vec{j} = (4 - 2 + 1)\vec{i} + (3 + 1 - 5)\vec{j} = 3\vec{i} - \vec{j}$

(ii) $2\vec{a} - 5\vec{b} = 2\left(4\vec{i} + 3\vec{j}\right) - 5\left(-2\vec{i} + \vec{j}\right) = 8\vec{i} + 6\vec{j} + 10\vec{i} - 5\vec{j} = (8 + 10)\vec{i} + (6 - 5)\vec{j} = 18\vec{i} + \vec{j}$

(iii) $\vec{a} + 2\vec{b} - 3\vec{c} = 4\vec{i} + 3\vec{j} + 2\left(-2\vec{i} + \vec{j}\right) - 3\left(\vec{i} - 5\vec{j}\right) = 4\vec{i} + 3\vec{j} - 4\vec{i} + 2\vec{j} - 3\vec{i} + 15\vec{j}$

$\Rightarrow \vec{a} + 2\vec{b} - 3\vec{c} = (4 - 4 - 3)\vec{i} + (3 + 2 + 15)\vec{j} = -3\vec{i} + 20\vec{j}$

Exercise 3A – vectors

1) List 5 vector quantities.
2) List 5 scalar quantities.
3) Which two of the following list of vectors are equal.
   $\vec{a} = 2\vec{i} + 3\vec{j} \qquad \vec{b} = 3\vec{i} + 2\vec{j} \qquad \vec{c} = -2\vec{i} - 3\vec{j} \qquad \vec{d} = -3\vec{i} - 2\vec{j} \qquad \vec{e} = 3\vec{i} + 2\vec{j}$

4) Which two of the following list of vectors are parallel.
$\vec{a} = 2\vec{i} - 2\vec{j}$ $\quad \vec{b} = -4\vec{i} + 4\vec{j}$ $\quad \vec{c} = 2\vec{i} + 2\vec{j}$ $\quad \vec{d} = -4\vec{i} - 4\vec{j}$ $\quad \vec{e} = 4\vec{i} + 4\vec{j}$

5) Which two of the following list of vectors have the same modulus (magnitude).
$\vec{a} = 3\vec{i} + 4\vec{j}$ $\quad \vec{b} = 2\vec{i} - 5\vec{j}$ $\quad \vec{c} = 6\vec{i} + 8\vec{j}$ $\quad \vec{d} = -4\vec{i} + 3\vec{j}$ $\quad \vec{e} = 3\vec{i} + 3\vec{j}$

6) Which two of the following list of vectors have equal magnitude and opposite directions.
$\vec{a} = 2\vec{i} + 3\vec{j}$ $\quad \vec{b} = -3\vec{i} + 2\vec{j}$ $\quad \vec{c} = -2\vec{i} + 3\vec{j}$ $\quad \vec{d} = 3\vec{i} - 2\vec{j}$ $\quad \vec{e} = 3\vec{i} + 2\vec{j}$

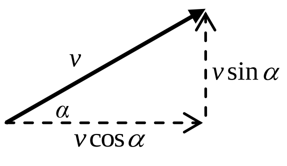
7) For each of the following values of tan α, write sin α and cos α as fractions, using surds where necessary.
(i) $\tan\alpha = \frac{3}{4}$, (ii) $\tan\alpha = \frac{12}{5}$, (iii) $\tan\alpha = \frac{7}{24}$, (iv) $\tan\alpha = \frac{15}{8}$, (v) $\tan\alpha = \frac{1}{2}$.

8) Put the following vectors into Cartesian form, where the $\vec{i}$ direction is East and the $\vec{j}$ direction is North.
(i) A displacement of 200 m in a direction 45° North of West.
(ii) A velocity of 40 m s^{-1} in a direction 60° South of West.
(iii) An acceleration of 6 m s^{-2} in a direction 30° North of East.
(iv) A force of 30 N in a direction $\tan^{-1}\frac{4}{3}$ South of East.
(v) A force of 39 N in a direction $\tan^{-1}\frac{12}{5}$ North of East.
(vi) A momentum of $10\sqrt{5}$ kg m s^{-1} in a direction $\tan^{-1}\frac{1}{2}$ South of West.

9) Find the magnitude and direction of the following vectors, where the $\vec{i}$ direction is East and the $\vec{j}$ direction is North.
(i) $5\vec{i} - 12\vec{j}$ m s^{-1}, (ii) $-12\vec{i} - 16\vec{j}$ m s^{-2}, (iii) $-9\vec{i} + 9\vec{j}$ kg m s^{-1},
(iv) $2\vec{i} + 4\vec{j}$ m, (v) $16\sqrt{3}\vec{i} - 16\vec{j}$ N, (vi) $-7\vec{i} + 21\vec{j}$ N.

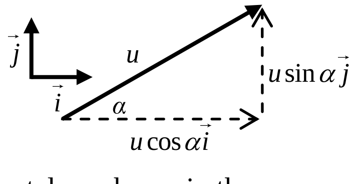
10) $\vec{a} = 2\vec{i} - 3\vec{j}$ $\quad \vec{b} = -2\vec{i} + 5\vec{j}$ $\quad \vec{c} = -3\vec{i} - 3\vec{j}$ $\quad \vec{d} = \vec{i} + 2\vec{j}$
Find the following resultants in $\vec{i} - \vec{j}$ form:
(i) $\vec{a} + \vec{b} + \vec{c} + \vec{d}$, (ii) $2\vec{b} - 3\vec{c}$, (iii) $-3\vec{a} + \vec{b} - 3\vec{c} + 2\vec{d}$, (iv) $-\vec{a} + 2\vec{d}$.

Section 3B: Projectiles on Level Horizontal Ground



Resolving vectors into horizontal and vertical components.
The diagram to the left shows how to break a vector (velocity) at angle α to the horizontal into horizontal and vertical components. The velocity v is the hypotenuse, the side adjacent to the angle is the *cos* component, and the side opposite the angle is the *sin* component.

The $\vec{i}$ - $\vec{j}$ Plane

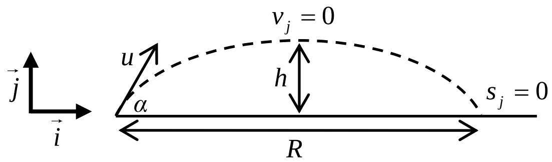


An easy way to represent vectors is to use the $\vec{i}$ - $\vec{j}$ plane.
$\vec{i}$ is a vector one unit long, along the x-axis, which is now called the $\vec{i}$ – axis.
$\vec{j}$ is a vector one unit long, along the y-axis, which is now called the $\vec{j}$ – axis.

If we have a projectile launched with an initial velocity u at an angle a to the horizontal, as shown in the diagram, then we write this initial velocity as a vector in $\vec{i}$ - $\vec{j}$ form as:

$$\vec{u} = u\cos a\vec{i} + u\sin a\vec{j}$$

For projectiles on the horizontal plane, u must always be written in this way.



If we now consider a projectile (such as a shot putt or a cannonball) projected in this way at some speed u at an angle α to the horizontal. We must consider the motion of such a projectile separately in the horizontal and vertical directions. The first step is to resolve the initial velocity into horizontal and vertical components as above: $\Rightarrow \vec{u} = u\cos a\vec{i} + u\sin a\vec{j}$

In the horizontal ($\vec{i}$) direction: As it is moving, it has **no acceleration** in the horizontal ($\vec{i}$) direction. This means it has constant velocity of $u\cos a$ in the $\vec{i}$ direction. The only formula we ever need to use in the $\vec{i}$ direction is: $s = ut$, to calculate the distance travelled in a certain time.

In the vertical ($\vec{j}$) direction: As it is moving, it has an acceleration of g (10 m s^{-2}) downwards in the vertical ($\vec{j}$) direction. The initial velocity in the $\vec{j}$ direction is $u\sin\alpha$. The velocity then decelerates to zero – at this stage it is at its **maximum height**, i.e. for maximum height $v_j = 0$. To get the maximum height then use $v^2 = u^2 + +2as$. After it has passed the maximum height point, it accelerates back towards the ground. At the point where it lands on the ground, the displacement in the $\vec{j}$ direction is zero. We use this to get the **time of flight** – use $s_j = 0$ in $s = ut + ½at^2$.

Using the time of flight, we can then calculate the **range** in the horizontal ($\vec{i}$) direction, and we can also calculate the **landing velocity**. Both of these are described in the solution strategy below. From the landing velocity, both the **landing speed** and the **landing angle** can be calculated. Another question sometimes asked is to find the **two times the projectile was at a particular height** during its trajectory.

Solution Strategy for projectiles on the horizontal plane questions:

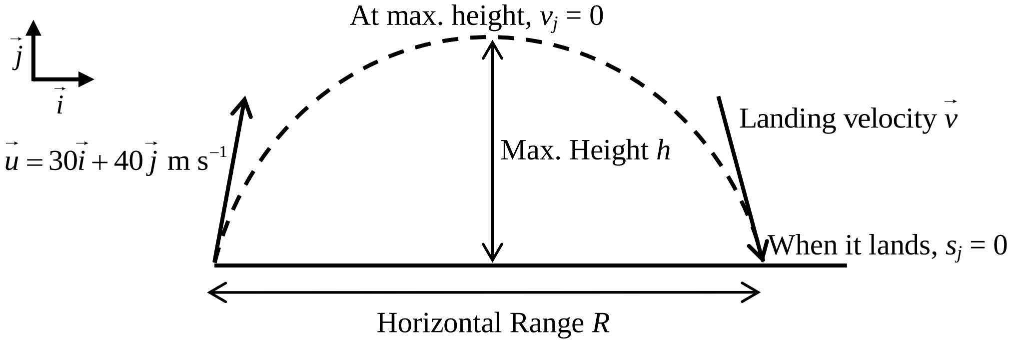
1) It is usually helpful to draw a **diagram**. The diagram is basically always the same, but it helps get the thinking straight on the question, and helps avoid mistakes.
2) Write the **initial velocity** in terms of $\vec{i}$ **and** $\vec{j}$, if it is not already given in $\vec{i}$ - $\vec{j}$ form.
3) At **maximum height,** $v_j = 0$. Use this in $v^2 = u^2 + 2as$ to find the max. height, or use $v = u + at$ to find the time at maximum height, both in the $\vec{j}$ direction.
4) Calculate the **time of flight** as follows. When the projectile lands, $s_j = 0$. Use this in the $\vec{j}$ direction in $s = ut + ½at^2$ to find t, the time of flight.
5) Use this t in $s = ut$ in the horizontal ($\vec{i}$) direction to find the **range**. N.B. There is no acceleration in the horizontal direction, so the velocity is constant.
6) To find the **landing velocity**, use the time of flight in $v = u + at$ in the $\vec{j}$ direction to find the $\vec{j}$ component. As there is no acceleration in the $\vec{i}$ direction, use the $\vec{i}$ component from the initial velocity. Together these give the landing velocity, $\vec{v}$, in $\vec{i}$ - $\vec{j}$ form. To find the **landing speed**, this is the magnitude of the landing velocity, $|\vec{v}|$. The **landing angle**, β, can be found from the landing velocity: $\tan\beta = \dfrac{-\vec{j}\text{ component}}{\vec{i}\text{ component}}$.
7) To find the **two times the projectile was at a particular height** during its trajectory, in the $\vec{j}$ direction, use $s = ut + ½at^2$ with s as the required height, and solve the quadratic equation for t.

Example 3B1

A particle is projected from a point on horizontal ground with an initial velocity of $30\vec{i}+40\vec{j}$ m s^{-1}.

Find: (i) its greatest height above the plane,
(ii) the time taken to reach the greatest height,
(iii) the time of flight of the particle,
(iv) the horizontal range of the particle,
(v) the speed of the particle as it strikes the ground,
(vi) the two times during its trajectory that it is at a height of 75 m.

Solution:



(i) For maximum height, $\vec{v}_j = 0$. In the $\vec{j}$ direction: $u = 40$ m s^{-1}, $a = -10$ m s^{-2}, $v = 0$, $s = h$

$$v^2 = u^2 + 2as, \qquad \Rightarrow 0 = 40^2 + 2(-10)h, \qquad \Rightarrow h = \frac{40^2}{20} = 80 \text{ m.}$$

Note: for maximum height problems, you can find the time it is at max. height first, as in part (ii), and then find the actual max. height using $s = ut + ½at^2$

(ii) In the $\vec{j}$ direction: $u = 40$ m s^{-1}, $a = -10$ m s^{-2}, $v = 0$, $t = ?$

$$v = u + at, \qquad \Rightarrow 0 = 40 - 10t, \qquad \Rightarrow t = \frac{40}{10} = 4 \text{ s.}$$

(iii) For time of flight, $s_j = 0$, $\Rightarrow$ In the $\vec{j}$ direction: $u = 40$ m s^{-1}, $a = -10$ m s^{-2}, $s = 0$, $t = ?$

$$s = ut + \tfrac{1}{2}at^2, \qquad \Rightarrow 0 = 40t + \tfrac{1}{2}(-10)t^2, \qquad \Rightarrow 0 = t(40 - 5t), \qquad \Rightarrow t = 0 \text{ or } t = \frac{40}{5} = 8 \text{ s.}$$

(iv) In the $\vec{i}$ direction: $u = 30$ m s^{-1}, $a = 0$, $t = 8$ s, $s = R$

$$s = ut, \qquad \Rightarrow R = 30(8) = 240 \text{ m.}$$

Note: for projectiles on flat horizontal ground, the time of flight is always twice as big as the time to reach the maximum height.

(v) As there is no acceleration in the $\vec{i}$ direction, the $\vec{i}$ component of $\vec{v}$ is the same as that of $\vec{u}$.

$\Rightarrow \vec{v} = 30\vec{i} + k\vec{j}$ m s^{-1}

In the $\vec{j}$ direction, use the time of flight in $v = u + at$. $u = 40$ m s^{-1}, $a = -10$ m s^{-2}, $t = 8$ s, $v = k$

$v = u + at, \qquad \Rightarrow k = 40 - 10(8) = -40$

$\Rightarrow \vec{v} = 30\vec{i} - 40\vec{j}$ m s^{-1}. This is the landing velocity.

The landing speed is the magnitude of the landing velocity: $|\vec{v}| = \sqrt{30^2 + (-40)^2} = 50$ m s^{-1}.

(vi) In the $\vec{j}$ direction: $u = 40$ m s^{-1}, $a = -10$ m s^{-2}, $s = 75$ m, $t = ?$

$$s = ut + \tfrac{1}{2}at^2, \qquad \Rightarrow 75 = 40t - 5t^2, \qquad \Rightarrow 5t^2 - 40t + 75 = 0$$

Divide across by 5, $\Rightarrow t^2 - 8t + 15 = 0$

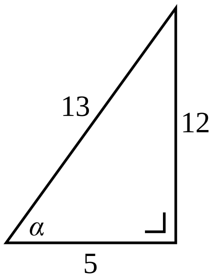
$\Rightarrow (t-3)(t-5) = 0, \qquad \Rightarrow t = 3$ s or $t = 5$ s.

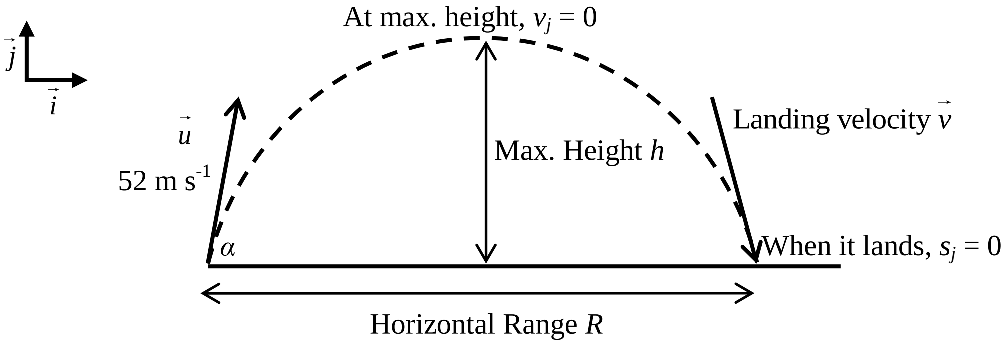
Example 3B2

A particle is projected from a point on horizontal ground with an initial speed of 52 m s^{-1} at an angle α to the horizontal, where $\tan\alpha = \frac{12}{5}$.

Find: (i) the initial velocity of the particle in terms of $\vec{i}$ and $\vec{j}$
(ii) the time taken to reach the greatest height,
(iii) its greatest height above the plane,
(iv) the horizontal range of the particle,
(v) the speed of the particle as it strikes the ground, and the angle it makes.

Solution:





(i) $\tan\alpha = \dfrac{12}{5}$, $\Rightarrow \sin\alpha = \dfrac{12}{13}$ and $\cos\alpha = \dfrac{5}{13}$.

$$\vec{u} = 52\cos\alpha\vec{i} + 52\sin\alpha\vec{j} = 52\left(\frac{5}{13}\right)\vec{i} + 52\left(\frac{12}{13}\right)\vec{j} = 20\vec{i} + 48\vec{j} \text{ m s}^{-1}$$

(ii) For maximum height, $\vec{v}_j = 0$. In the $\vec{j}$ direction: $u = 48$ m s^{-1}, $a = -10$ m s^{-2}, $v = 0$, $t = ?$

$$v = u + at, \qquad \Rightarrow 0 = 48 - 10t, \qquad \Rightarrow t = \frac{48}{10} = 4.8 \text{ s.}$$

(iii) In the $\vec{j}$ direction: $u = 48$ m s^{-1}, $a = -10$ m s^{-2}, $v = 0$, $s = h$

$$v^2 = u^2 + 2as, \qquad \Rightarrow 0 = 48^2 + 2(-10)h, \qquad \Rightarrow h = \frac{48^2}{20} = 115.2 \text{ m.}$$

(iv) For time of flight, $\vec{s}_j = 0$, $\Rightarrow$ In the $\vec{j}$ direction: $u = 48$ m s^{-1}, $a = -10$ m s^{-2}, $s = 0$, $t = ?$

$$s = ut + \tfrac{1}{2}at^2, \qquad \Rightarrow 0 = 48t + \tfrac{1}{2}(-10)t^2, \qquad \Rightarrow 0 = t(48 - 5t), \qquad \Rightarrow t = 0 \text{ or } t = \frac{48}{5} = 9.6 \text{ s.}$$

For range, in the $\vec{i}$ direction: $u = 20$ m s^{-1}, $a = 0$, $t = 9.6$ s, $s = R$

$$s = ut, \qquad \Rightarrow R = 20(9.6) = 192 \text{ m.}$$

(v) As there is no acceleration in the $\vec{i}$ direction, the $\vec{i}$ component of $\vec{v}$ is the same as that of $\vec{u}$.

$$\Rightarrow \vec{v} = 20\vec{i} + k\vec{j} \text{ m s}^{-1}$$

In the $\vec{j}$ direction, use the time of flight in $v = u + at$. $u = 48$ m s^{-1}, $a = -10$ m s^{-2}, $t = 9.6$ s, $v = k$

$$v = u + at, \qquad \Rightarrow k = 48 - 10(9.6) = -48$$

$\Rightarrow \vec{v} = 20\vec{i} - 48\vec{j}$ m s^{-1}. This is the landing velocity.

The landing speed is the magnitude of the landing velocity: $\left|\vec{v}\right| = \sqrt{20^2 + (-48)^2} = 52$ m s^{-1}.

The landing angle, β, is found from: $\tan\beta = \dfrac{-v_j}{v_i} = \dfrac{-(-48)}{20} = \dfrac{48}{20} = \dfrac{12}{5}$

$$\Rightarrow \beta = \tan^{-1}\frac{12}{5} = 67.4^\circ$$

Exercise 3B – projectiles on the horizontal plane

1) A particle is projected from a point on horizontal ground with an initial velocity of $25\vec{i}+30\vec{j}$ m s^{-1}.

Find: (i) its greatest height above the plane,
(ii) the time taken to reach the greatest height,
(iii) the time of flight of the particle,
(iv) the horizontal range of the particle,
(v) the speed of the particle as it strikes the ground,
(vi) the two times during its trajectory that it is at a height of 33.75 m.

2) A particle is projected from a point on horizontal ground with an initial velocity of $20\vec{i}+15\vec{j}$ m s^{-1}.

Find: (i) its greatest height above the plane,
(ii) the time taken to reach the greatest height,
(iii) the time of flight of the particle,
(iv) the horizontal range of the particle,
(v) the speed of the particle as it strikes the ground,
(vi) the two times during its trajectory that it is at a height of 10 m.

3) A particle is projected from a point on horizontal ground with an initial velocity of $16\vec{i}+30\vec{j}$ m s^{-1}.

Find: (i) the time taken to reach the greatest height,
(ii) its greatest height above the plane,
(iii) the speed of the particle when it is at its greatest height,
(iv) the horizontal range of the particle.

4) A particle is projected from a point on horizontal ground with an initial speed of 50 m s^{-1} at an angle α to the horizontal, where $\tan\alpha = \frac{4}{3}$.

Find: (i) the initial velocity of the particle in terms of $\vec{i}$ and $\vec{j}$
(ii) the time taken to reach the greatest height,
(iii) its greatest height above the plane,
(iv) the horizontal range of the particle,
(v) the speed of the particle as it strikes the ground, and the angle it makes.

5) A particle is projected from a point on horizontal ground with an initial speed of 68 m s^{-1} at an angle α to the horizontal, where $\tan\alpha = \frac{15}{8}$.

Find: (i) the initial velocity of the particle in terms of $\vec{i}$ and $\vec{j}$
(ii) its greatest height above the plane,
(iii) the time of flight of the particle,
(iv) the horizontal range of the particle,
(v) the speed of the particle as it strikes the ground, and the angle it makes,
(vi) the two times the particle is at a height of 160 m.

6) A particle is projected from a point on horizontal ground with an initial speed of 50 m s^{-1} at an angle α to the horizontal, where $\tan\alpha = \frac{7}{24}$.

Find: (i) the initial velocity of the particle in terms of $\vec{i}$ and $\vec{j}$
(ii) its greatest height above the plane,
(iii) the time of flight of the particle,
(iv) the horizontal range of the particle,
(v) the speed of the particle as it strikes the ground, and the angle it makes,
(vi) the two times the particle is at a height of 9 m.

7) A particle is projected from a point o with an initial speed of $20\sqrt{2}$ m s^{-1} at an angle of θ above the horizontal. It strikes the level ground at point p after 4 seconds.

(i) Find the angle θ.

(ii) Find $|op|$, the horizontal range of the particle from o to p.

8) A golf ball is struck from a point o with an initial speed of 40 m s^{-1} at an angle of θ above the horizontal. It strikes the level ground at point p after 4 seconds.
(i) Find the angle θ.
(ii) Write the initial velocity of the golf ball in terms of $\vec{i}$ and $\vec{j}$.
(iii) Find $|op|$, the horizontal range of the golf ball from o to p, correct to the nearest metre.

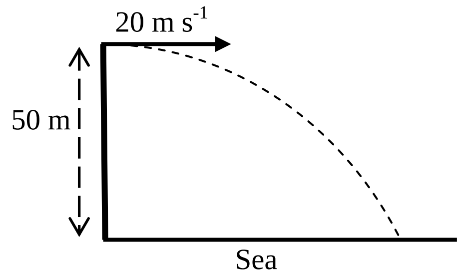
Section 3C: Projectiles from Cliffs and Buildings

In section 3B we saw projectiles landing on the same horizontal level as they were projected, and we were able to find the time of flight and the horizontal range. In this section we will concentrate on projectiles which are being fired from a height, maybe the top of a cliff or building, and finding the time of flight for them to land on the ground at the bottom, and also the horizontal range. Particular care must be taken to find the time of flight in this case, as is demonstrated in the examples.

Solution Strategy for projectiles on the horizontal plane projected from a height:

1) It is usually helpful to draw a **diagram**. The diagram is basically always the same, but it helps get the thinking straight on the question, and helps avoid mistakes.
2) Write the **initial velocity** in terms of $\vec{i}$ **and** $\vec{j}$, if it is not already given in $\vec{i}$ - $\vec{j}$ form. In a number of these questions, the particle is projected horizontally, in which case the initial velocity has no $\vec{j}$ component.
3) At **maximum height,** $v_j = 0$. Use this in $v^2 = u^2 + 2as$ to find the max. height, or use $v = u + at$ to find the time at maximum height, both in the $\vec{j}$ direction.
4) If the particle is projected horizontally, then find the **time of flight** using $s = ut + ½at^2$ in the $\vec{j}$ direction, with u as zero and the acceleration, g, positive downwards. If the particle is projected at an angle, then there are two different ways the time of flight can be calculated. First of all find the time for the particle to reach its maximum height, and then find the time for it to freefall all the way from this height to the ground, and add these two time together. Otherwise when the projectile lands, $s_j = -h$, where h is the height of the cliff. Use this in the $\vec{j}$ direction in $s = ut + ½at^2$ to find t, the time of flight.
5) Use this t in $s = ut$ in the horizontal ($\vec{i}$) direction to find the **range**. N.B. There is no acceleration in the horizontal direction, so the velocity is constant.

Example 3C1
A particle is projected horizontally from the top of a straight vertical cliff of height 50 m high, with a speed of 20 m s^{-1}. How far from the foot of the cliff will it hit the sea?



Solution: $\vec{u} = 20\vec{i}$ m s^{-1}

To find the time of flight, in the $\vec{j}$ direction use: $s = ut + \frac{1}{2}at^2$

$\Rightarrow 50 = 0t + \frac{1}{2}(10)t^2$

$\Rightarrow 50 = 5t^2$

$\Rightarrow t = \sqrt{10} = 3.16$ s

To get range, in the $\vec{i}$ direction, $s = ut$

$\Rightarrow R = 20(3.16) = 63.2$ m.

Example 3C2

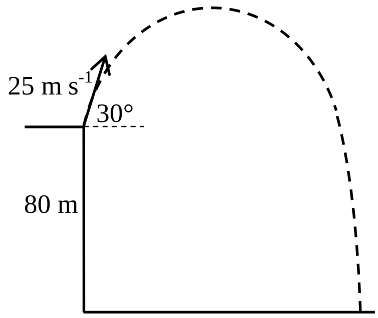
A particle is projected from the top of a seaside cliff 80 m high, with a speed of 25 m s^{-1} at an angle of 30° above the horizontal. Find

(i) the maximum height reached above the water by the particle,

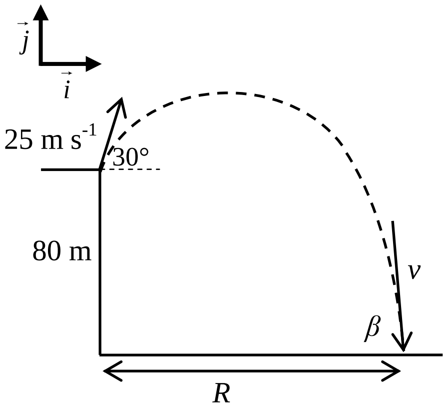
(ii) the time of flight,

(iii) the horizontal distance from the foot of the cliff to where the particle lands in the water,

(iv) the speed of the particle as it hits the water, and the angle it makes.



Solution:



(i) $\vec{u} = 25\cos 30^\circ \vec{i} + 25\sin 30^\circ \vec{j} = 12.5\sqrt{3}\,\vec{i} + 12.5\vec{j}$

For maximum height, in the $\vec{j}$ direction, $\vec{v}_j = 0$

$u = 12.5$ m s^{-1}, $a = -10$ m s^{-2}, $v = 0$, $s = ?$

$v^2 = u^2 + 2as$

$\Rightarrow 0^2 = 12.5^2 - 2(10)s$

$\Rightarrow s = \dfrac{156.25}{20} = 7.8$ m

$\Rightarrow$ The maximum height above the water is 80 + 7.8 = 87.8 m.

(ii) To find the time of flight, $\vec{s}_j = -80$: $s = ut + \frac{1}{2}at^2$

$\Rightarrow -80 = 12.5t - 5t^2$

$\Rightarrow 5t^2 - 12.5t - 80 = 0$

$\Rightarrow t = \dfrac{12.5 \pm \sqrt{12.5^2 - 4(5)(-80)}}{2(5)} = \dfrac{12.5 \pm 41.91}{10}$

$\Rightarrow t = 5.44$ s or $t =$ negative

(iii) To get range, in the $\vec{i}$ direction, $s = ut$

$\Rightarrow R = 12.5\sqrt{3}(5.44) = 117.8$ m.

(iv) To find $\vec{v}$, first of all there is no change in the $\vec{i}$ direction,

$\Rightarrow \vec{v} = 12.5\sqrt{3}\,\vec{i} + p\vec{j}$

To find p, in the $\vec{j}$ direction, $v = u + at$,

$\Rightarrow p = 12.5 - 10(5.44) = -41.9$

$\Rightarrow \vec{v} = 12.5\sqrt{3}\,\vec{i} - 41.9\vec{j}$ m s^{-1}

The speed, $|\vec{v}| = \sqrt{\left(12.5\sqrt{3}\right)^2 + (-41.9)^2} = 47.2$ m s^{-1}

The landing angle, $\beta = \tan^{-1} - \dfrac{-41.9}{12.5\sqrt{3}} = 62.7^\circ$

(ii) An alternative method to find the time of flight. First find the time taken to reach maximum height from the point of projection:

$u = 12.5$ m s^{-1}, $a = -10$ m s^{-2}, $v = 0$, $t = ?$

$v = u + at$

$\Rightarrow 0 = 12.5 - 10t$, $\Rightarrow t_1 = 1.25$ s

Now find the time from the maximum height down to the sea freefalling,

$u = 0$, $a = 10$ m s^{-2}, $s = 87.8$ m, $t = ?$

$s = ut + \frac{1}{2}at^2$

$\Rightarrow 87.8 = 0t + \frac{1}{2}(10)t^2$

$\Rightarrow 5t^2 = 87.8$

$\Rightarrow t_2 = \sqrt{\dfrac{87.8}{5}} = 4.19$ s

Time of flight $= t_1 + t_2 = 1.25 + 4.19 = 5.44$ s

Example 3C3

A straight vertical cliff is 45 m high. Projectile P is fired horizontally directly out to sea from the top of the cliff with a speed of u m s^{-1}. Projectile P hits the sea at a distance of 60 m from the foot of the cliff.

(i) Find the time it takes projectile P to hit the sea.

(ii) Find the value of u.

Another projectile, Q, is fired upwards at an angle α to the horizontal and with initial speed 20 m s^{-1} directly out to sea from the top of the cliff. Projectile Q takes two seconds longer than projectile P to hit the sea.

(iii) Show that $\sin\alpha = \frac{4}{5}$.

(iv) How far from the foot of the cliff does projectile Q hit the sea?

Solution: (i) In the $\vec{j}$ direction use: $s = ut + \frac{1}{2}at^2$

$u = 0,\ \ a = 10 \text{ m s}^{-2},\ \ s = 45 \text{ m},\ t = ?$

$\Rightarrow 45 = 0t + \frac{1}{2}(10)t^2, \qquad \Rightarrow 45 = 5t^2, \qquad \Rightarrow t = \sqrt{9} = 3 \text{ s}$

(ii) In the $\vec{i}$ direction use: $s = ut$

$u = u,\ \ s = 60 \text{ m},\ t = 3 \text{ s}$

$\Rightarrow 60 = 3u, \qquad \Rightarrow u = 20 \text{ m s}^{-1}$

(iii) $t = 3 + 2 = 5$ s.

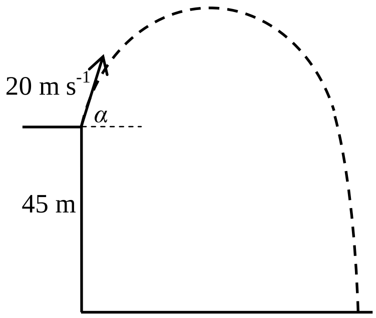
$\vec{u} = 20\cos\alpha\vec{i} + 20\sin\alpha\vec{j}$ m s^{-1}

In the $\vec{j}$ direction use: $s = ut + \frac{1}{2}at^2$

$u = 20\sin\alpha \text{ m s}^{-1},\ \ a = -10 \text{ m s}^{-2},\ \ t = 5 \text{ s},\ \ s = -45 \text{ m},$

$\Rightarrow -45 = 20\sin\alpha(5) - 5(5)^2, \qquad \Rightarrow -45 = 100\sin\alpha - 125,$

$\Rightarrow 80 = 100\sin\alpha, \qquad \Rightarrow \sin\alpha = \frac{80}{100} = \frac{4}{5}$ Q.E.D.



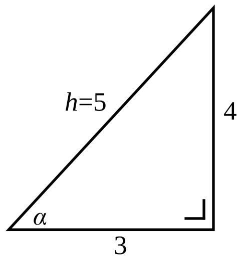
(iv) $\sin\alpha = \frac{4}{5}, \qquad \Rightarrow \cos\alpha = \frac{3}{5}$

$\Rightarrow \vec{u} = 20\cos\alpha\vec{i} + 20\sin\alpha\vec{j} = 20\left(\frac{3}{5}\right)\vec{i} + 20\left(\frac{4}{5}\right)\vec{j} = 12\vec{i} + 16\vec{j}$ m s^{-1}

To get range, in the $\vec{i}$ direction use: $s = ut$

$u = 12 \text{ m s}^{-1},\ t = 5 \text{ s},\ \ s = ?$

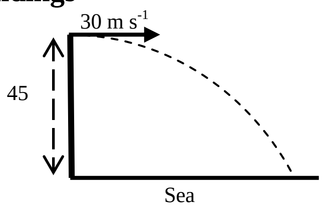
$\Rightarrow s = 12(5) = 60$ m.



Exercise 3C – projectiles on the horizontal plane fired from cliffs and buildings

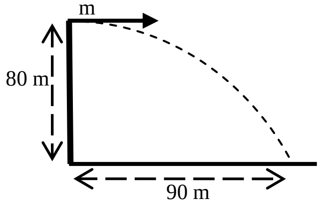
1) A particle is projected horizontally from the top of a straight vertical cliff of height 45 m with an initial speed of 30 m s^{-1}.

How far from the foot of the cliff will the particle hit the sea?



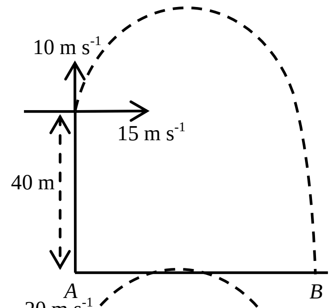
2) A straight vertical cliff is 80 m high. A projectile is fired horizontally with an initial speed of u m s^{-1} from the top of the cliff. It strikes the level ground at a distance of 90 m from the foot of the cliff.

Find the value of u.

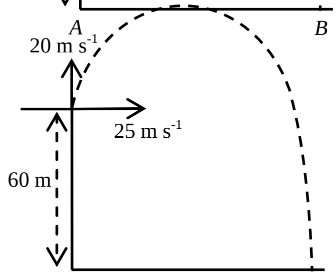


3) A smooth rectangular box is fixed to the horizontal ground. A ball is moving with constant speed u m s-1 on the top of the box. The ball is moving parallel to the side of the box. The ball rolls a distance of 1.5 m in 0.5 seconds, before falling over an edge of the box.
   (i) Find the value of u.
   (ii) The ball strikes the horizontal ground at a distance of $\frac{3}{2\sqrt{5}}$ m from the bottom of the box. Find the height of the box.

4) A particle is projected from the top of a straight vertical building of height 40 m with velocity $15\vec{i}+10\vec{j}$ m s^{-1}. It strikes the horizontal ground at *B*.
   Find: (i) the maximum height above the ground level,
   (ii) the time of flight,
   (iii) |*AB*|, the distance from *A* to *B*,
   (iv) the speed of the particle as it strikes the ground at *B*.



5) A projectile is fired with initial velocity $25\vec{i}+20\vec{j}$ m s^{-1} from the top of a vertical cliff of height 60 m.
   (i) Calculate the time taken to reach the maximum height.
   (ii) Calculate the maximum height fo the projectile above ground level.
   (iii) Calculate the time of flight.
   (iv) Find the range.
   (v) Find the speed of the projectile as it strikes the ground.



6) A projectile is fired upwards from the top of a cliff of height 25 m with a velocity of 25 m s^{-1} at an angle α to the horizontal, where $\tan\alpha = 4/3$.
The projectile strikes the sea at a distance *d* from the foot of the cliff. Calculate *d*.

7) (i) A straight vertical cliff is 45 m high. Projectile P is fired horizontally directly out to sea from the top of the cliff with a speed of *u* m s^{-1}. Projectile P hits the sea at a distance of 75 m from the foot of the cliff. Find the time it takes projectile P to hit the sea, and find the value of *u*.
(ii) Another projectile, Q, is fired upwards at an angle α to the horizontal and with an initial speed of 25 m s^{-1} directly out to sea from the cliff. Projectile Q takes two seconds longer than projectile P to hit the sea. Show that $\sin\alpha = \frac{16}{25}$, and find how far from the foot of the cliff projectile Q hits the sea.

8) (i) A vertical building is 20 m high. Projectile P is fired horizontally directly from the top of the building with speed 30 m s^{-1}. How long does it take projectile P to hit the horizontal ground? At what distance from the bottom of the building does the projectile hit the ground?
(ii) Projectile Q is also fired from the top of the same building, with initial velocity $x\vec{i}+y\vec{j}$ m s^{-1}.
Projectile Q takes three times as long to hit the ground as projectile P did, and projectile Q hits the horizontal ground twice as far from the bottom of the building as projectile P did. Show that the value of *x* is 20, and find the value of *y*, correct to one place of decimals.

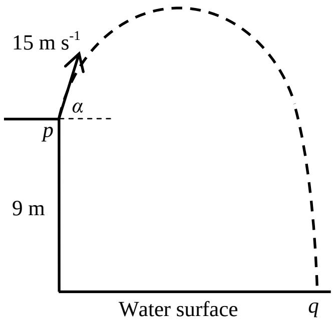
9) (i) Two missiles, P and Q, are fired simultaneously out to sea from the top of a cliff 40 m above sea level. Missile P is fired horizontally with a speed of 30 m s^{-1}. Missile Q is fired upwards with a speed of 30 m s^{-1} at an angle of 45° to the horizontal. Each missile strikes the sea. Show that the time of flight of missile Q is twice that of missile P.
(ii) Calculate the distance between the two points at which the missiles strike the sea, correct to one place of decimals.

10) A projectile is fired upwards from a point p with an initial speed of 15 m s^{-1} inclined at an angle α to the horizontal where $\tan\alpha = \frac{4}{3}$. The point p is 9 m above the surface of the water in a pool. The projectile strikes the water at point q.

Find: (i) the time the projectile takes to reach point q,

(ii) the velocity of the projectile at q in terms of $\vec{i}$ and $\vec{j}$,

(iii) the angle between the path of the projectile and the horizontal as it enters the water at q. Give your answer correct to the nearest degree.



Section 3D: Projectiles Passing through Specific Points on the Trajectory

Where a projectile passes through a specific point on its path, the $\vec{i}$ and $\vec{j}$ directions must be dealt with separately. In the $\vec{i}$ direction, an expression for the time can be found by using $s = ut$. This expression for the time should then be used in the $\vec{j}$ direction in
$s = ut + ½at^2$.

Solution Strategy for questions involving hitting targets:

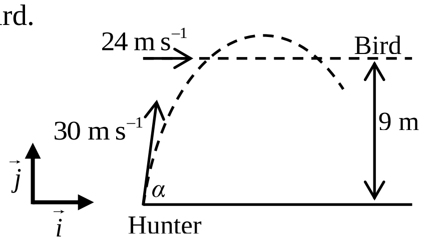
1) Given a distance in the **horizontal direction ($\vec{i}$)**, use $\boldsymbol{s = ut}$ to find the time it is at that distance.
2) Given a height in the **vertical direction ($\vec{j}$)**, use $\boldsymbol{s = ut + ½at^2}$ to get two times when it is at that height, the smaller one on the way up and the larger one on the way back down.
3) To find the **velocity** at a particular time in the flight, the $\vec{i}$ component is the same as for $\vec{u}$. In the $\vec{j}$ direction use $\boldsymbol{v = u + at}$ to find the $\vec{j}$ component. Use Pythagoras' Theorem then to find the magnitude of the velocity (speed), and the tan of the angle to find the direction.

Example 3D1

A bird flies out of a tree at a height of 9 m above a hunter's gun, flying horizontally at 24 m s^{-1}. At the same instant the hunter fires a bullet at a speed of 30 m s^{-1} in the direction of the bird.

(i) Find the angle of projection of the bullet if it is to hit the bird.

(ii) Find the time taken for the bullet to hit the bird.

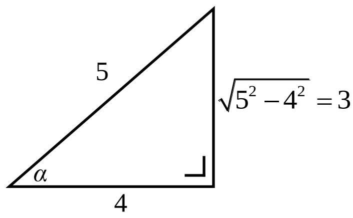


Solution: (i) For the bullet, $\vec{u} = 30\cos\alpha\vec{i} + 30\sin\alpha\vec{j}$ m s^{-1}

For the bullet to hit the bird, their velocities in the $\vec{i}$ direction must be equal, $\Rightarrow 30\cos\alpha = 24$,

$\Rightarrow \cos\alpha = \frac{24}{30} = \frac{4}{5}$

$\Rightarrow$ the angle of projection is $\cos^{-1}\frac{4}{5} = 36.9^\circ$



$\sin\alpha = \frac{3}{5}$

(ii) For the bullet $\vec{u} = 30\cos\alpha\vec{i} + 30\sin\alpha\vec{j} = 24\vec{i} + 18\vec{j}$ m s^{-1}

In the $\vec{j}$ direction, $s = ut + \frac{1}{2}at^2$, $\Rightarrow 9 = 18t - 5t^2$, $\Rightarrow 5t^2 - 18t + 9 = 0$,

$\Rightarrow (5t - 3)(t - 3) = 0$, $\Rightarrow t = 0.6$ s or 3 s.

Therefore the bullet hits the bird after 0.6 s as this happens first.

Example 3D2

A particle is projected from a point o on level horizontal ground with an initial speed of 52 m s^{-1} at an angle θ to the horizontal, where $\tan\theta = \frac{5}{12}$.

(i) Find the initial velocity of the particle in terms of $\vec{i}$ and $\vec{j}$.

After 3 seconds in flight the particle hits a target p which is above the ground.

(ii) Show that the distance from the point o to the target p is 145 m, to the nearest metre.

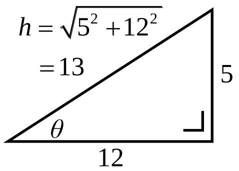
(iii) How far below the highest point reached by the particle is the target p?

(iv) Find, correct to the nearest m s^{-1}, the speed of the particle as it hits the target at p.

Solution:

$h = \sqrt{5^2 + 12^2}$

$= 13$



$\sin\theta = \dfrac{5}{13}$

$\cos\theta = \dfrac{12}{13}$

(i) $\vec{u} = 52\cos\theta\vec{i} + 52\sin\theta\vec{j} = 52\left(\dfrac{12}{13}\right)\vec{i} + 52\left(\dfrac{5}{13}\right)\vec{j} = 48\vec{i} + 20\vec{j}$ m s^{-1}

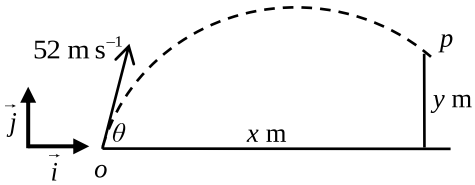
(ii) In the $\vec{i}$ direction: $s = ut$

$\Rightarrow x = 48(3) = 144$ m

In the $\vec{j}$ direction: $s = ut + \frac{1}{2}at^2$

$\Rightarrow y = 20(3) - 5(3)^2 = 15$ m

$\Rightarrow |op| = \sqrt{144^2 + 15^2} = 144.8 \text{ m} \approx 145$ m



(iii) At max. height: $v_j = 0$, $\quad v^2 = u^2 + 2as$

$\Rightarrow 0^2 = 20^2 + 2(-10)h$, $\quad \Rightarrow$ max. height $h = 20$ m

$\Rightarrow$ the target is 5 m below the highest point.

(iv) $\vec{v} = 48\vec{i} + k\vec{j}$ m s^{-1}. In the direction: $v = u + at$, $\quad \Rightarrow k = 20 - 10(3) = -10$ m s^{-1}

$\Rightarrow \vec{v} = 48\vec{i} - 10\vec{j}$ m s^{-1}, $\quad \Rightarrow |\vec{v}| = \sqrt{48^2 + (-10)^2} = 49$ m s^{-1}

Exercise 3D – projectiles passing through specific points on their trajectory

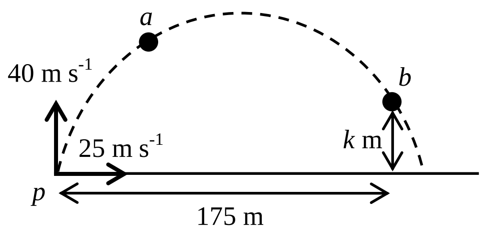
1) A particle is projected with initial velocity $25\vec{i} + k\vec{j}$ m s^{-1} from a point on a horizontal plane. When its horizontal displacement is 75 m, it is 12 m above the plane. Find the value of k.

2) A plane is flying horizontally with a speed of 120 m s^{-1}. As it flies at a height of 300 m over a gun, a shell is fired at a speed of 200 m s^{-1}. Find the angle of projection of the shell so that it hits the plane, and find the time it takes to hit the plane.

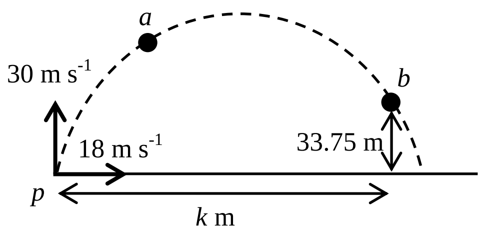
3) A duck is flying horizontally with a speed of 25 m s^{-1}. As it flies over an archer at a height of 55 m, the archer fires an arrow with a speed of 65 m s^{-1}. Find the angle of projection of the arrow if it hits the duck.

4) A particle is projected with initial velocity $25\vec{i} + 40\vec{j}$ m s^{-1} from point p on a horizontal plane. a and b are two points on the trajectory (path) of the particle. The particle reaches point a after 3 seconds of motion. The displacement of point b from p is $175\vec{i} + k\vec{j}$ metres.

Find: (i) the velocity of the particle at a in terms of $\vec{i}$ and $\vec{j}$,
(ii) the speed and direction of the particle at a,
(iii) the value of k.

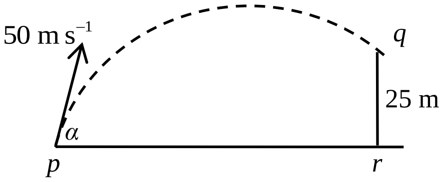


5) A particle is projected with initial velocity $18\vec{i} + 30\vec{j}$ m s^{-1} from point p on a horizontal plane. a and b are two points on the trajectory (path) of the particle. The particle reaches point a after 2 seconds of motion. The displacement of point b from p is $k\vec{i} + 33.75\vec{j}$ metres.



Find: (i) the velocity of the particle at *a* in terms of $\vec{i}$ and $\vec{j}$,
(ii) the speed and direction of the particle at *a*,
(iii) the value of *k*.

6) A golfer hits a ball from a point *p* on level horizontal ground with an initial speed of 50 m s^{-1} inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. As the ball is descending it strikes a tree at a point *q*, where $|qr| = 25$ m and *qr* is perpendicular to *pr*.



(i) Find the initial velocity of the particle in terms of $\vec{i}$ and $\vec{j}$.
(ii) Find the time taken for the ball to reach *q*.
(iii) Calculate $|pr|$.

7) A particle is projected from a point *p* on level horizontal ground with an initial speed of 25 m s^{-1} at an angle θ to the horizontal, where $\tan\theta = \frac{4}{3}$.
(i) Find the initial velocity of the particle in terms of $\vec{i}$ and $\vec{j}$.
After 3 seconds in flight the particle hits a target *q* which is above the ground.
(ii) Show that the distance from the point *p* to the target *q* is $15\sqrt{10}$ m.
(iii) How far below the highest point reached by the particle is the target *p*?
(iv) Find, correct to the nearest m s^{-1}, the speed of the particle as it hits the target at *p*.

Answers to Exercises

Exercise 3A

3) $\vec{b}$ & $\vec{e}$. 4) $\vec{c}$ & $\vec{e}$. 5) $\vec{a}$ & $\vec{d}$. 6) $\vec{b}$ & $\vec{d}$. 7) (i) $\frac{3}{5}, \frac{4}{5}$, (ii) $\frac{12}{13}, \frac{5}{13}$, (iii) $\frac{7}{25}, \frac{24}{25}$, (iv) $\frac{15}{17}, \frac{8}{17}$, (v) $\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}$.

8) (i) $-100\sqrt{2}\vec{i}+100\sqrt{2}\vec{j}$ m, (ii) $-20\vec{i}-20\sqrt{3}\vec{j}$ m s^{-1}, (iii) $3\sqrt{3}\vec{i}+3\vec{j}$ m s^{-2}, (iv) $18\vec{i}-24\vec{j}$ N, (v) $15\vec{i}+36\vec{j}$ N,

8) (vi) $-20\vec{i}-10\vec{j}$ kg m s^{-1}. 9) (i) 13 m s^{-1} 67.4° S of E, (ii) 20 m s^{-2} 53.1° S of W, (iii) $9\sqrt{2}$ kg m s^{-1} 45° N of W,

9) (iv) $2\sqrt{5}$ m 63.4° N of E, (v) 32 N 30° S of E, (vi) $7\sqrt{10}$ N 71.6° N of W.

10) (i) $-2\vec{i}+\vec{j}$, (ii) $5\vec{i}+19\vec{j}$, (iii) $3\vec{i}+27\vec{j}$, (iv) $7\vec{j}$.

Exercise 3B

1) (i) 45 m, (ii) 3 s, (iii) 6 s, (iv) 150 m, (v) 39.1 m s^{-1}, (vi) 1.5 s, 4.5 s.
2) (i) 11.25 m, (ii) 1.5 s, (iii) 3 s, (iv) 60 m, (v) 25 m s^{-1}, (vi) 1 s, 2 s.
3) (i) 3 s, (ii) 45 m, (iii) 16 m s^{-1}, (iv) 96 m.
4) (i) $30\vec{i}+40\vec{j}$ m s^{-1}, (ii) 4 s, (iii) 80 m, (iv) 240 m, (v) 50 m s^{-1}, 53.1°.
5) (i) $32\vec{i}+60\vec{j}$ m s^{-1}, (ii) 180 m, (iii) 12 s, (iv) 384 m, (v) 68 m s^{-1}, 61.9°, (vi) 4 s, 8 s.
6) (i) $48\vec{i}+14\vec{j}$ m s^{-1}, (ii) 9.8 m, (iii) 2.8 s, (iv) 134.4 m, (v) 50 m s^{-1}, 16.3°, (vi) 1 s, 1.8 s.
7) (i) 45°, (ii) 80 m. 8) (i) 60°, (ii) $20\sqrt{3}\vec{i}+20\vec{j}$ m s^{-1}, (iii) 139 m.

Exercise 3C

1) 90 m. 2) 22.5 m s^{-1}. 3) (i) 3 m s^{-1}, (ii) 0.25 m. 4) (i) 45 m, (ii) 4 s, (iii) 60 m, (iv) 33.5 m s^{-1}.
5) (i) 2 s, (ii) 20 m, (iii) 6 s, (iv) 150 m, (v) 47.2 m s^{-1}. 6) 75 m. 7) (i) 3 s, 25 m s^{-1}, (ii) 96.0 m.
8) (i) 2 s, 60 m, (ii) 26.7 m $^{-1}$. 9) (ii) 35.1 m. 10) (i) 3 s, (ii) $9\vec{i}-18\vec{j}$ m s^{-1}, (iii) 63°.

Exercise 3D

1) 19 m s^{-1}. 2) 53.1°, 2 s. 3) 67.4°, 1 s. 4) (i) $25\vec{i}+10\vec{j}$ m s^{-1}, (ii) 26.9 m s^{-1}, 21.8°,
4) (iii) 35. 5) (i) $18\vec{i}+10\vec{j}$ m s^{-1}, (ii) 20.6 m s^{-1}, 29.1°, (iii) 81.
6) (i) $40\vec{i}+30\vec{j}$ m s^{-1}, (ii) 5 s, (iii) 200 m. 7) (i) $15\vec{i}+20\vec{j}$ m s^{-1}, (iii) 5 m, (iv) 18 m s^{-1}.