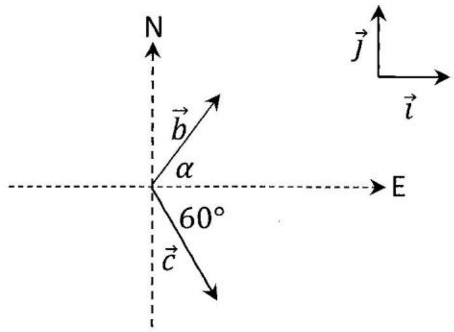
Dominick Donnelly's solutions to the
Leaving Certificate Applied Mathematics Ordinary Level Sample Paper 2022

Question 1

(a) A displacement vector, $\vec{b}$, has a magnitude of 15 km and a direction α north of east, where $\tan\alpha = \frac{4}{3}$.

A second displacement vector, $\vec{c}$, has a magnitude of $10\sqrt{3}$ km and a direction 60° south of east, as shown in the diagram.



(i) Express $\vec{b}$ and $\vec{c}$ in terms of the unit vectors $\vec{i}$ and $\vec{j}$.

$\tan\alpha = \frac{4}{3}$ $\quad \sin\alpha = \frac{4}{5}$ $\quad \cos\alpha = \frac{3}{5}$

$\underset{\sim}{b} = 15\cos\alpha\,\underset{\sim}{i} + 15\sin\alpha\,\underset{\sim}{j} = 9\underset{\sim}{i} + 12\underset{\sim}{j}$ km.

$\underset{\sim}{c} = 10\sqrt{3}\cos 60°\,\underset{\sim}{i} - 10\sqrt{3}\sin 60°\,\underset{\sim}{j} = 5\sqrt{3}\,\underset{\sim}{i} + 15\underset{\sim}{j}$ km.

(ii) Calculate $\vec{b}\cdot\vec{c}$, the dot product of $\vec{b}$ and $\vec{c}$.

$\underset{\sim}{b}\cdot\underset{\sim}{c} = (9\underset{\sim}{i} + 12\underset{\sim}{j})\cdot(5\sqrt{3}\,\underset{\sim}{i} + 15\underset{\sim}{j})$

$= 45\sqrt{3} + 180 = \underline{257.9}$

A third displacement vector, $\vec{d}$, is perpendicular to $\vec{b}$.
$\vec{d} = -4\vec{i} + k\vec{j}$.

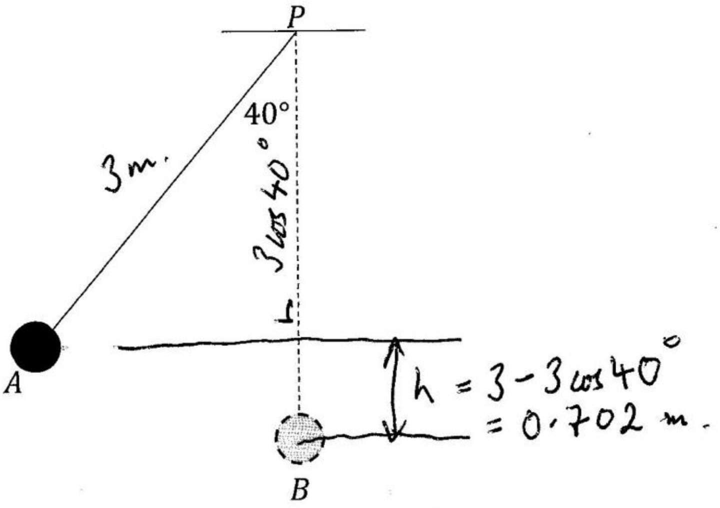
(iii) Calculate k.

$\underset{\sim}{b}\cdot\underset{\sim}{d} = (9\underset{\sim}{i} + 12\underset{\sim}{j})\cdot(-4\underset{\sim}{i} + k\underset{\sim}{j}) = 0$

$\Rightarrow -36 + 12k = 0$

$\Rightarrow \underline{k = 3}$

(b) A small smooth sphere of mass 2 kg is connected by a light inextensible string of length 3 m to a fixed point P. The sphere is held at position A, where the taut string makes an angle of 40° to the vertical, as shown in the diagram. The sphere is then released from rest.



(i) The motion of the sphere may be modelled using the principle of conservation of energy. Using this model, calculate the speed of the sphere as it passes through position B, when the string is vertical.

P.E. at A = K.E at B

$$mgh = \frac{1}{2}mv^2$$

$$2g(0.702) = \frac{1}{2}(2)v^2 \quad \Rightarrow v = \sqrt{13.76} = \underline{3.71 \text{ m s}^{-1}}$$

(ii) Calculate the centripetal force on the sphere as it passes through B.

$\uparrow T$, (2kg), $\uparrow a$, $\downarrow 2g$

$$a = \frac{v^2}{r} = \frac{3.71^2}{3} = 4.59 \text{ m s}^{-2}$$

Centripetal force = Resultant force

$$F = ma \quad \Rightarrow F = 2(4.59) = \underline{9.17 \text{ N.}}$$

(iii) Calculate the tension in the string when the sphere passes through B.

$$T - 2g = F = \text{resultant force} = 9.17$$

$$\Rightarrow T = 9.17 + 2g = \underline{28.8 \text{ N.}}$$

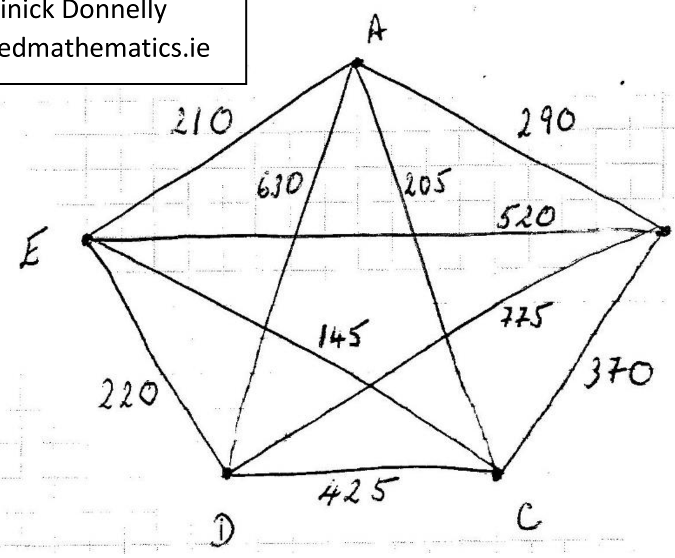
Question 2

(a) During a treasure hunt competition, Seán must search at each of locations A, B, C, D and E. He may start at whichever of these location he chooses and he may visit the other locations in any order.

The estimated time, in seconds, needed to travel between any two of these locations is shown in the following table.

Time (s)	A	B	C	D	E
A	–	290	205	630	210
B	290	–	370	775	520
C	205	370	–	425	145
D	630	775	425	–	220
E	210	520	145	220	–

(i) Draw a network to represent this information. On your network the weights of the edges should represent the times to travel between the locations, which should be represented by labelled nodes.

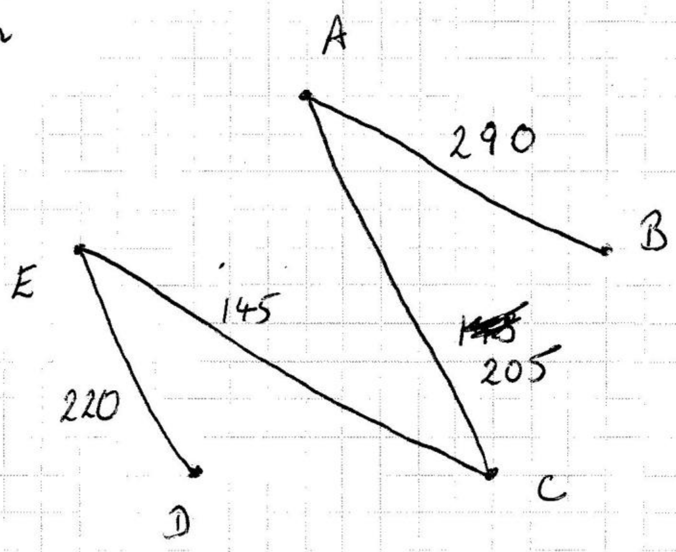


In order to win the competition, Seán wants to spend as little time as possible travelling between the locations.

(ii) Using an appropriate algorithm, find the minimum spanning tree for this network. Name the algorithm you used. Relevant supporting work must be shown.

Kruskal's Algorithm

CE	145	✓
AC	205	✓
AE	210	✗
DE	220	✓
AB	290	✓
BC	370	✗
CD	425	✗
BE	520	✗
AD	630	✗
BD	775	✗



$290 + 205 + 220 + 145 = \underline{860\text{ s}}$

(iii) At which location should Seán start? Justify your answer.

B or D He visits, in order: BACED or in reverse.

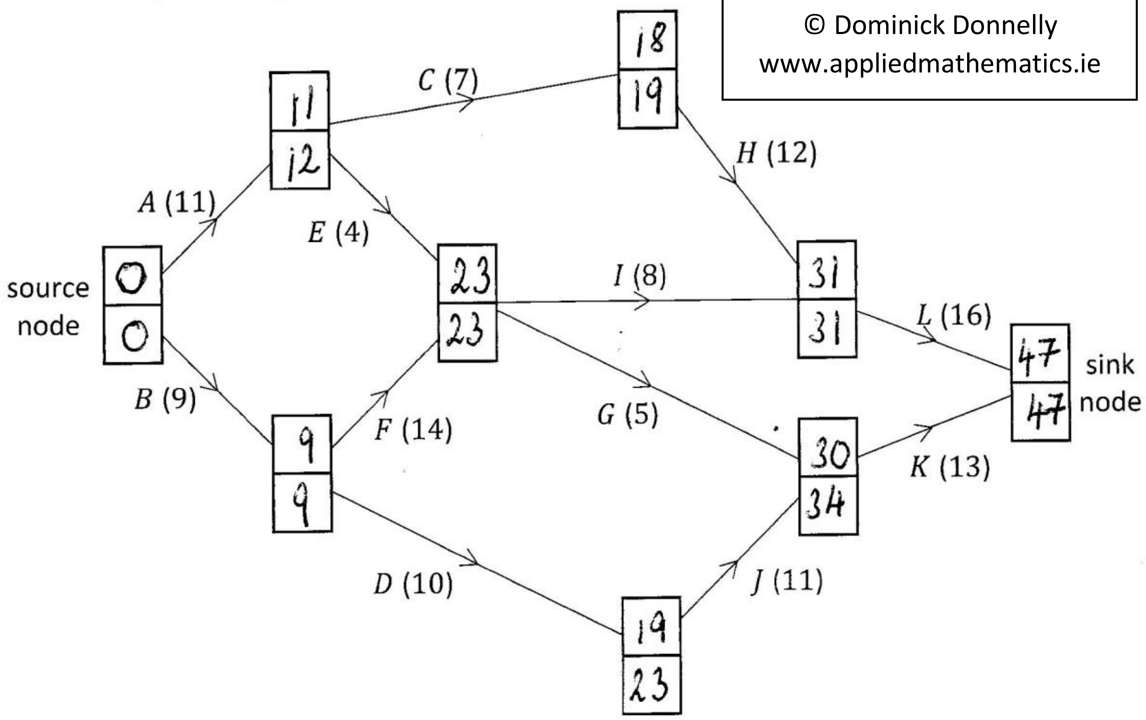
(b) The diagram below shows the scheduling network used in the assembly of an air filtering system.

The edges of the network represent the activities that have to be completed as part of the assembly and are labelled with the letters A to L. The letters used to label the edges should **not** be taken as representing the order in which the activities happen. The time, in minutes, to complete each of the activities is shown.

The nodes of the network represent events or points in time during the assembly.
The source node is the time when the project begins and the sink node is the time when the project ends.

(i) Calculate the early time and the late time for each event.

Complete the diagram below by writing the early time (upper box) and late time (lower box) at the node representing each event.



(ii) Write down the critical path for the network.

BFIL

(iii) Write down the minimum time, in minutes, needed to assemble an air filtering system.

47 minutes.

(iv) Select any one non-critical activity on the network and calculate its float, in minutes.

D : 23 − 10 − 9 = 4 minutes.

Question 3

(a) Kate wishes to invest €150 000 in a long-term investment scheme. Cormac is an investment broker. He offers Kate a guaranteed annual interest rate of 5.2% on her investment. However Cormac will charge an annual fee of €3000, which will be deducted from her investment.

The value, P, in €, of Kate's investment after n years may be modelled by the difference equation:

$$P_{n+1} = 1.052P_n - 3000$$

where $n \geq 0, n \in \mathbb{Z}$ and $P_0 = 150\,000$.

(i) Solve this difference equation to find an expression for P_n, the value of Kate's investment after n years if she invests with Cormac.

$P_0 = 150,000 \quad \Rightarrow P_1 = 1.052(150,000) - 3000 =$
$= 154,800$

General Solution: $P_n = A(1.052)^n + B$

$P_0: \quad 150,000 = A + B \quad (1)$

$P_1: \quad 154,800 = 1.052A + B \quad (2)$

$(2)-(1): \quad 0.052A = 4,800 \quad \Rightarrow A = 92,308$

In (1): $B = 150,000 - 92,308 = 57,692$

$\Rightarrow P_n = 92,308(1.052)^n + 57,692$

(ii) Calculate P_6, the value of Kate's investment after 6 years if she invests with Cormac.

$P_6 = 92,308(1.052)^6 + 57,692 = €182,814$

(iii) Ruth, another investment broker, offers Kate a guaranteed annual interest rate of 4.3%. Ruth will charge an annual fee of €2000.

Kate wishes to maximise the value of her investment after 6 years. With which broker, Cormac or Ruth, should Kate invest? Justify your answer.

$P_{n+1} = 1.043P_n - 2000$

$P_0 = 150,000 \quad P_1 = 1.043(150,000) - 2000$
$= 154,450$

General Solution: $P_n = A(1.043)^n + B$

$P_0: \quad 150,000 = A + B \quad (1)$

$P_1: \quad 154,450 = 1.043A + B \quad (2)$

$(2)-(1): \quad 4450 = 0.043A \quad \Rightarrow A = 103,488$

In (1): $B = 150,000 - 103,488 = 46,512$

$\Rightarrow P_n = 103,488(1.043)^n + 46,512$

$P_6 = 103,488(1.043)^6 + 46,512 = €179,740$ (Ruth)

therefore investing with Cormac is better.

(b) A car dealership began to sell a new type of electric car in January 2020. The dealership sold eight of these cars in 2020. It sold twelve of them in 2021.

A sales person predicts that U, the number of such cars sold in any year, will be equal to twice the number of cars sold in the previous year plus three times the number of cars sold the year before that.

This prediction can be expressed as the second-order difference equation:

$$U_{n+2} - 2U_{n+1} - 3U_n = 0$$

where $n \geq 0, n \in \mathbb{Z}, U_0 = 8$ and $U_1 = 12$.

This difference equation has the characteristic quadratic equation $x^2 - 2x - 3 = 0$.

(i) Solve this quadratic equation, i.e. calculate the two roots of the equation.

~~Characteristic~~ $x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$\Rightarrow x = -1$ or $x = 3$

(ii) Hence or otherwise, solve the difference equation to find an expression for U_n in terms of n.

$U_n = p(-1)^n + q(3)^n$

$U_0 = 8 \Rightarrow 8 = p + q$ (1)

$U_1 = 12 \Rightarrow 12 = -p + 3q$ (2)

(1) + (2): $20 = 4q \Rightarrow q = 5$

In (1): $p = 8 - 5 = 3$

$\Rightarrow U_n = 3(-1)^n + 5(3)^n$

(iii) Calculate the number of such cars that the model predicts the dealership will sell between the start of 2020 and the end of 2025.

$U_5 = 3(-1)^5 + 5(3)^5 = -3 + 1215 = 1212$

$U_4 = 3(-1)^4 + 5(3)^4 = 3 + 405 = 408$

$U_3 = 3(-1)^3 + 5(3)^3 = -3 + 135 = 132$

$U_2 = 3(-1)^2 + 5(3)^2 = 3 + 45 = 48$

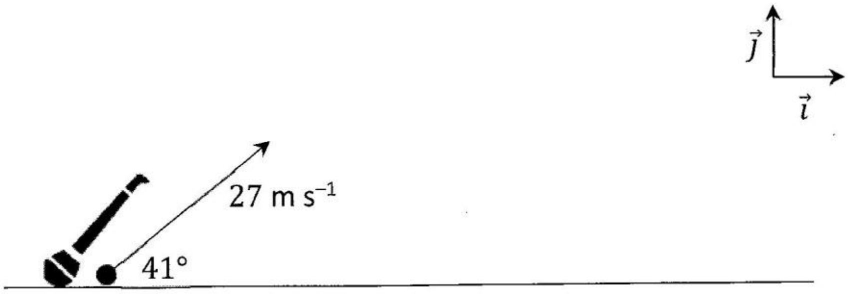
$U_1 = \quad = 12$

$U_0 = \quad = 8 \; +$

1820 cars.

Question 4

A camogie player strikes a sliotar off the horizontal ground so that it travels with an initial velocity of 27 m s^{-1} at an angle of 41° to the ground, as shown in the diagram.



(i) Express the initial velocity of the sliotar in terms of the unit vectors $\vec{i}$ and $\vec{j}$.

$\underset{\sim}{u} = 27 \cos 41^\circ \underset{\sim}{i} + 27 \sin 41^\circ \underset{\sim}{j}$

$\underset{\sim}{u} = 20.38 \underset{\sim}{i} + 17.71 \underset{\sim}{j} \quad m\,s^{-1}$

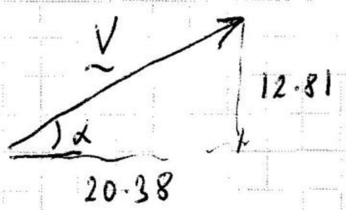
The motion of the sliotar may be modelled as projectile motion in a vertical plane, ignoring the effects of wind and the effects of air resistance.

(ii) Calculate the speed and direction of the sliotar 0.5 s after it is struck.

$\underset{\sim}{j}: v = u + at$

$\Rightarrow V_j = 17.71 - 9.8(0.5) = 12.81$

$\Rightarrow \underset{\sim}{V} = 20.38 \underset{\sim}{i} + 12.81 \underset{\sim}{j} \quad m\,s^{-1}$



$|\underset{\sim}{V}| = \sqrt{20.38^2 + 12.81^2} = \underline{24.1 \; m\,s^{-1}}$

$\tan\alpha = \dfrac{12.81}{20.38} \qquad \underline{\alpha = 32.2^\circ}$

(iii) Calculate the time it takes for the sliotar to reach its maximum height.

At max. height: $V_j = 0$

$v = u + at$

$0 = 17.71 - 9.8t$

$\Rightarrow t = \dfrac{17.71}{9.8} = \underline{1.81 \text{ s.}}$

(iv) Calculate the maximum height of the sliotar.

$$v^2 = u^2 + 2as$$

$$0 = 17.71^2 - 2(9.8)h$$

$$\Rightarrow h = \frac{17.71^2}{19.6} = \underline{16.0 \text{ m.}}$$

(v) The crossbar in a camogie goal is 2.5 m above the ground. Calculate the time interval during which the sliotar is at least 2.5 m above the ground.

$$\underset{\sim}{j}: \quad s_j = 2.5 \text{ m.}$$

$$s = ut + \tfrac{1}{2}at^2$$

$$2.5 = 17.71t - 4.9t^2$$

$$\Rightarrow 4.9t^2 - 17.71t + 2.5 = 0$$

$$\Rightarrow t = \frac{17.71 \pm \sqrt{17.71^2 - 4(4.9)(2.5)}}{9.8}$$

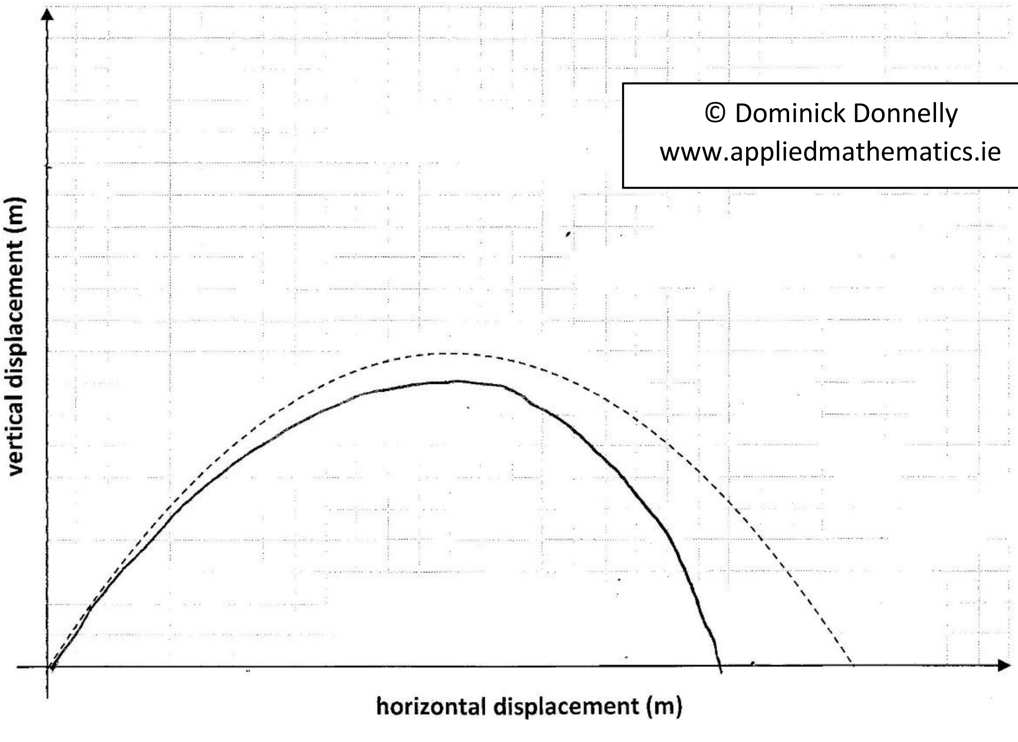
$$\Rightarrow t = \frac{17.71 \pm 16.27}{9.8}$$

$$\Rightarrow t = 0.15 \text{ s.} \quad \text{or } t = 3.47 \text{ s.}$$

$$\Rightarrow \text{time interval from } \underline{0.15 \text{ s}} \rightarrow \underline{3.47 \text{ s.}}$$

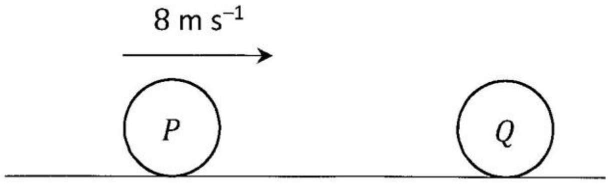
(vi) The graph below shows the predicted path of the sliotar when the effects of wind and the effects of air resistance are ignored. The graph is not drawn to scale.

Using the same axes, sketch the path you would expect the sliotar to take if the model took into account the effects of air resistance (but not the effects of wind).



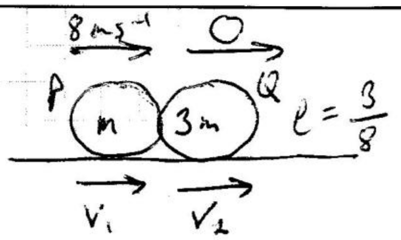
Question 5

(a) A small smooth sphere, P, of mass m, travels along a horizontal surface at a constant speed of 8 m s^{-1}. It collides with another small smooth sphere, Q, of mass $3m$, which is at rest.



The coefficient of restitution between the spheres is $\frac{3}{8}$.

(i) Calculate the velocity of P and the velocity of Q after impact.



$m_1 u_1 + m_2 u_2 = m_1 V_1 + m_2 V_2$

$8\cancel{m} + 0 = \cancel{m} V_1 + 3\cancel{m} V_2 \quad \Rightarrow 8 = V_1 + 3V_2$ (1)

$-e = \frac{V_2 - V_1}{u_2 - u_1} \quad -\frac{3}{8} = \frac{V_2 - V_1}{0 - 8} \quad \Rightarrow 3 = V_2 - V_1$ (2)

(1) + (2): $4V_2 = 11 \quad \Rightarrow V_2 = \frac{11}{4}$ m s^{-1}

In (2): $V_1 = V_2 - 3 = \frac{11}{4} - 3 = -\frac{1}{4}$ m s^{-1}

(ii) Calculate, in terms of m, the loss in kinetic energy due to the impact.

K.E. before $= \frac{1}{2} m (8)^2 = 32m$ J.

K.E. after $= \frac{1}{2} m \left(-\frac{1}{4}\right)^2 + \frac{1}{2} 3m \left(\frac{11}{4}\right)^2$

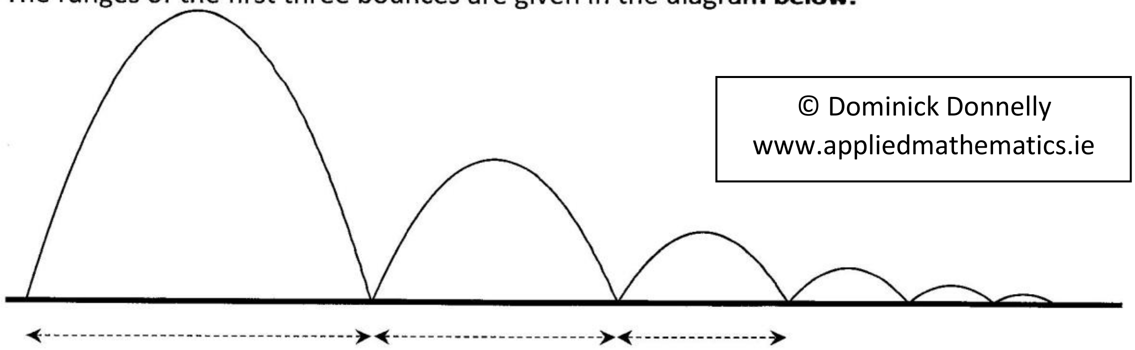
$= \frac{m}{32} + \frac{3m}{2}\left(\frac{121}{16}\right)$

$= \frac{m}{32} + \frac{363m}{32} = \frac{364m}{32} = \frac{91m}{8}$ J.

Loss of K.E. $= 32m - \frac{91m}{8} = \frac{165m}{8}$ J.

(b) A tennis ball bounces across a tennis court. It is found that some of the ball's kinetic energy is lost each time it hits the ground, such that the horizontal range, R, of each bounce is **70%** of the range of the previous bounce.

The ranges of the first three bounces are given in the diagram **below.**



This geometric sequence may be represented by the difference equation:

$$R_{n+1} = 0.7R_n$$

where $n \geq 0, n \in \mathbb{Z}$ and $R_0 = 160$ cm.

(i) Solve this difference equation to find an expression for R_n in terms of n.

General Solution : $R_n = A(0.7)^n + B$

$R_0 = 160 \Rightarrow 160 = A + B$ ①

$R_1 = 112 \Rightarrow 112 = 0.7A + B$ ②.

① − ② : $48 = 0.3A \Rightarrow A = 160$

In ①: $B = 160 - 160 = 0$

$\Rightarrow R_n = 160(0.7)^n$

(ii) Calculate R_6 in cm, to two decimal places.

$R_6 = 160(0.7)^6 = \underline{18.82 \text{ cm}}$.

(iii) Calculate S_6, the sum of the ranges of the first seven bounces, in cm, to two decimal places.

$a = 160 \quad r = 0.7 \quad S_n = \dfrac{a(1-r^n)}{1-r}$

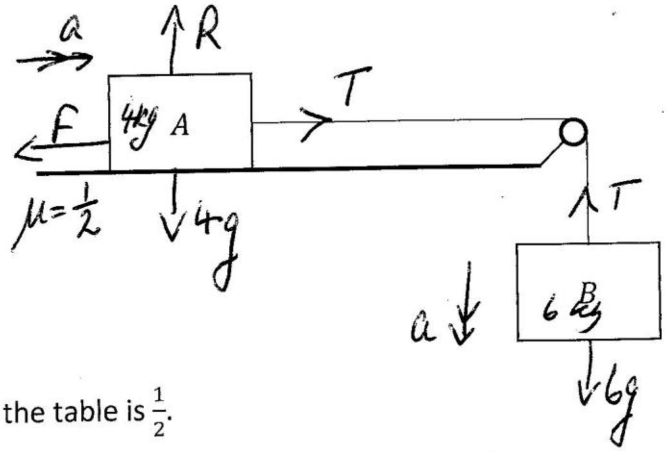
$\Rightarrow S_7 = \dfrac{160(1-0.7^7)}{1-0.7} = \underline{489.41 \text{ cm}}.$

(iv) Write a difference equation for the horizontal ranges of the bounces if no kinetic energy is lost when the ball hits the ground.

If no K.E. lost $\Rightarrow e = 1 \quad \Rightarrow R_0 = R_1 = R_2 = \cdot$ etc.

$\Rightarrow R_{n+1} = \underline{1}\, R_n$.

Question 6

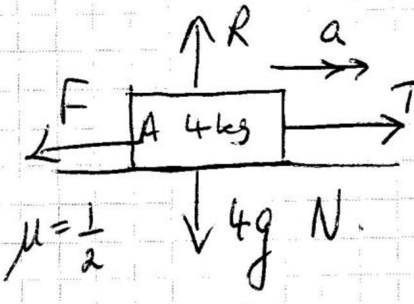


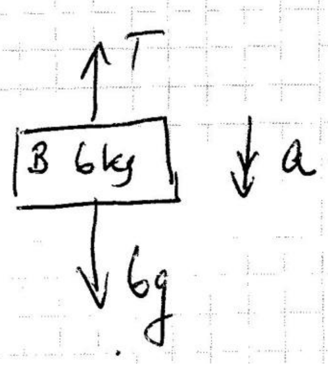
Block A, of mass 4 kg, rests on a rough horizontal table. It is connected to block B, of mass 6 kg, by a light inextensible string which passes over a fixed smooth pulley at the edge of the table.

When the system is released from rest, block A is 40 cm from the pulley.

The coefficient of friction between block A and the table is $\frac{1}{2}$.

(i) Draw diagrams to show the forces acting on blocks A and B while they are moving.





(ii) Calculate the frictional force acting on block A while it is moving.

Resolve ↑ : $R = 4g$ N.

$F = \mu R = \frac{1}{2}(4g) = 2g = \underline{19.6 \text{ N}}$.

(iii) Calculate the tension in the string and the acceleration of the blocks while they are moving.

$F = ma$.

4 kg →: $T - 19.6 = 4a$ (1)

6 kg ↓ : $6g - T = 6a$ (2)

(1) + (2): $4g = 10a \Rightarrow a = \frac{4g}{10} = \frac{2g}{5} = \underline{3.92 \text{ ms}^{-2}}$

In (1): $T = 4(3.92) + 19.6 = \underline{35.28} \text{ N}$.

(iv) Calculate the speed of block A when it reaches the pulley.

$u = 0,\ a = 3.92\ m s^{-2},\ s = 40\ cm = 0.4\ m,\ v = ?$

$$v^2 = u^2 + 2as$$

$$v^2 = 0 + 2(3.92)(0.4)$$

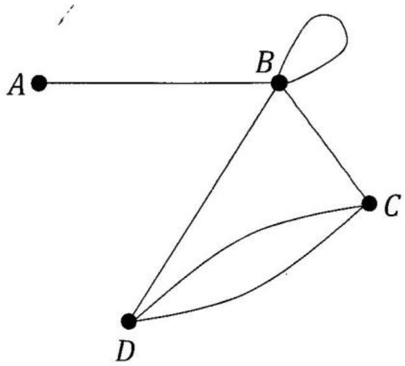
$$\Rightarrow v = 1.77\ m s^{-1}$$

(v) Explain why it would not be appropriate to model this problem using the principle of conservation of energy.

There is work done in overcoming the friction force, so this would have to be included in the energy calculation, not just potential and kinetic energy.

Question 7

(a) Write the adjacency matrix that represents the graph shown below.



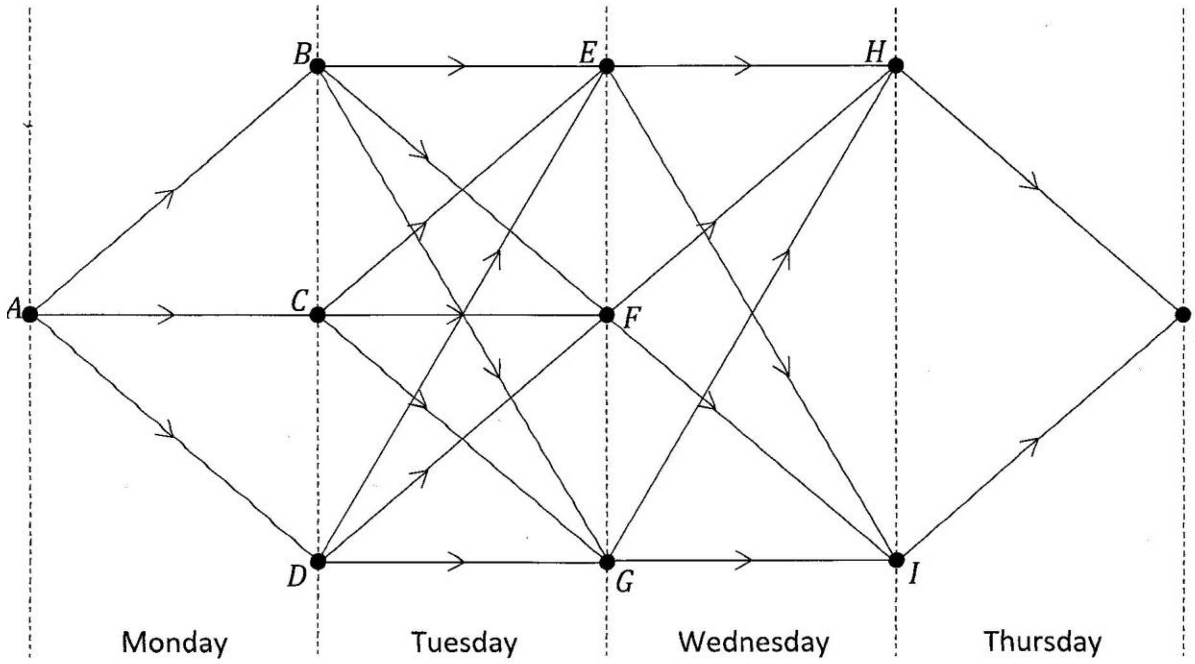
$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix} \end{array}$$

(b) Matrix $P = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Matrix $Q = \begin{bmatrix} 3 & 0 \\ -2 & 2 \end{bmatrix}$. Calculate PQ.

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 2\times 3 + 1(-2) & 2(0) + 1(2) \\ 0(3) + 1(-2) & 0(0) + 1(+2) \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -2 & 2 \end{pmatrix}$$

A coach operator wishes to design a new four-day coach route from city A to city J.
The coach will depart from city A on Monday morning and should arrive in city J on Thursday evening. On Monday night the coach will stop in city B, C or D. On Tuesday night the coach will stop in city E, F or G. On Wednesday night the coach will stop in city H or I.
Passengers may begin or end their journey at any city.

The operator draws the network shown below to help him design this route.



The table below gives the number of passengers who wish to travel between pairs of cities on each day.

Journey	Number of passengers	Journey	Number of passengers
A to B	32	D to F	45
A to C	27	D to G	23
A to D	19	E to H	43
B to E	36	E to I	34
B to F	41	F to H	17
B to G	45	F to I	26
C to E	22	G to H	32
C to F	38	G to I	46
C to G	29	H to J	36
D to E	30	I to J	25

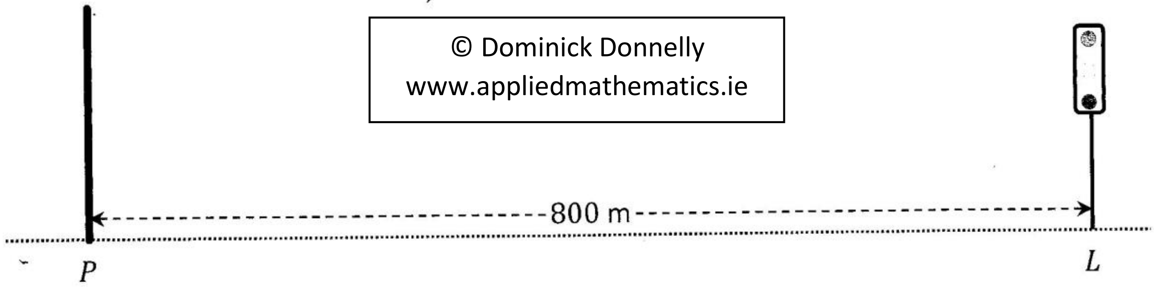
Use Bellman's Principle of Optimality to calculate the path from city A to city J which maximises the number of passengers who use the coach. Relevant supporting work must be shown.

Stage	State	Action	Destination	Value
1 Thursday	H	HJ	J	36 *
	I	IJ	J	25 *
2 Weds.	E	EH	H	43 + 36 = 79 *
		EI	I	34 + 25 = 59
	F	FH	H	17 + 36 = 53 *
		FI	I	26 + 25 = 51
	G	GH	H	32 + 36 = 68
		GI	I	46 + 25 = 71 *
3 Tuesday	B	BE	E	36 + 79 = 115
		BF	F	41 + 53 = 94
		BG	G	45 + 71 = 116 *
	C	CE	E	22 + 79 = 101 *
		CF	F	38 + 53 = 91
		CG	G	29 + 71 = 100
	D	DE	E	30 + 79 = 109 *
		DF	F	45 + 53 = 98
		DG	G	23 + 71 = 94
4 Monday	A	AB	B	32 + 116 = 148 *
		AC	C	27 + 101 = 128
		AD	D	19 + 109 = 128

Optimal Policy : ABGIJ Optimal Value: 148 passengers.

Question 8

Pole P and traffic lights L lie 800 m apart on a straight level road, as in the diagram below.



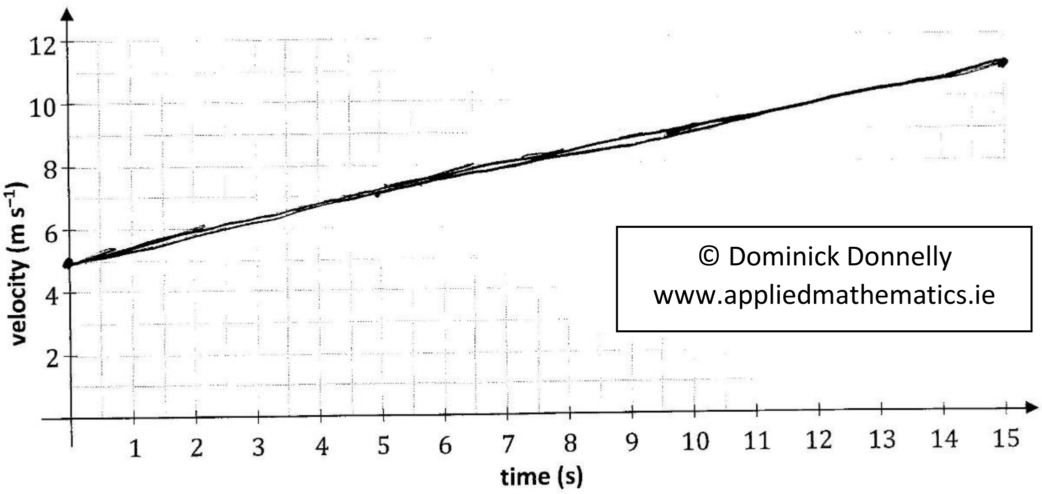
A car passes P travelling towards L with a speed of 5 m s^{-1} and an acceleration of 0.4 m s^{-2}. At the same moment, a motorcycle passes L travelling towards P with a speed of 4 m s^{-1} and an acceleration of 0.6 m s^{-2}.

(i) Calculate the speed of the car 15 s after it passes P.

$v = u + at$

$v = 5 + 0.4(15) = \underline{11 \text{ m s}^{-1}}$

(ii) Draw a velocity-time graph for the motion of the car for the first 15 s after it passes P.



(iii) Write an expression for $s_c(t)$, the displacement of the car from P at any time t.

$$s = ut + \tfrac{1}{2}at^2$$
$$s_c = 5t + 0.2t^2 \quad \text{m.}$$

(iv) Write an expression for $s_m(t)$, the displacement of the motorcycle from L at any time t.

$$s = ut + \tfrac{1}{2}at^2$$
$$s_m = 4t + 0.3t^2. \quad \text{m.}$$

(v) The car and the motorcycle pass each other after T seconds. Calculate T.

$$s_c + s_m = 800$$
$$\Rightarrow 5t + 0.2t^2 + 4t + 0.3t^2 = 800$$
$$\Rightarrow 0.5t^2 + 9t - 800 = 0$$
$$\Rightarrow t^2 + 18t - 1600 = 0$$
$$\Rightarrow (t+50)(t-32) = 0$$
$$\Rightarrow \cancel{t = -50} \quad \text{or} \quad \underline{t = 32\text{s}.}$$

At the instant that the car and motorcycle pass each other, the car stops accelerating and continues travelling at the velocity it has at that instant.

(vi) Calculate the total time it takes the car to travel from P to L.

$$v = u + at \quad = 5 + 0.4(32) = 17.8 \text{ m s}^{-1}$$
$$s = ut + \tfrac{1}{2}at^2 \quad = 5(32) + 0.2(32)^2 = 364.8 \text{ m}$$
$$\Rightarrow \text{Distance to go} = 800 - 364.8 = 435.2 \text{ m.}$$
$$s = ut \quad \Rightarrow t = \frac{s}{u} = \frac{435.2}{17.8} = 24.45 \text{ s.}$$

$$\Rightarrow \text{Total time} = 32 + 24.45 = \underline{56.45 \text{ s.}}$$